Linear algebra

Introduction

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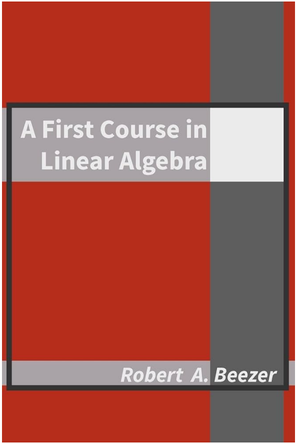
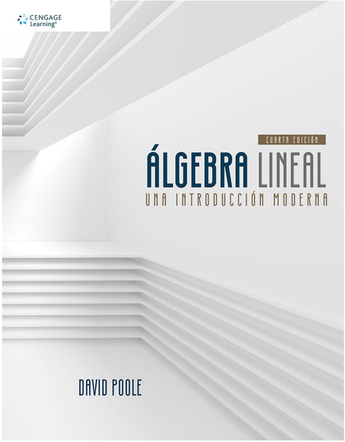
Bibliography





http://linear.ups.edu/

Linear algebra

Linear

Linear : (Intuitively) Anything that is "straight" of "flat". For example, in the xy-plane you might be accustomed to describing straight lines (is there any other kind?) as the set of solutions to an equation of the form $y = mx + b$, where the slope m and the y-intercept b are constants that together describe the line. In three-dimensions, with coordinates described by triples (x, y, z), planes can be described as the set of solutions to equations of the form $ax + by + cz = d$, where a,b,c,d are constants that together determine the plane.

Linear

Linear : Another view of this notion of “flatness” is to recognize that the sets of points just described are solutions to equations of a relatively simple form. These equations involve **addition** and **multiplication** only. Here are some examples of typical equations we will see in the next slides:

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$$2x+3y-4z=13 \qquad 4x_1+5x_2-x_3+x_4+x_5=0 \qquad 9a-2b+7c+2d=-7$$

Linear

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$$2x+3y-4z=13 \qquad 4x_1+5x_2-x_3+x_4+x_5=0 \qquad 9a-2b+7c+2d=-7$$

What we will not see are equations like:

$$xy+5yz=13 \qquad x_1+x_2^3/x_4-x_3x_4x_5^3=0 \qquad tan(ab)+log(c-d)=-7$$

Algebra

The word “algebra” is derived from the Arabic word *al-gebr*, meaning reunion of broken parts. During the 11th century, it was perhaps the Islamic world that represented the most mathematically sophisticated civilization. However, there was no algebraic manipulation of the kind seen in modern texts, and medieval mathematical writing was rethorical, with everything being described in words. This “algebra” is the algebra of real numbers, which for millenia was explicitly defined as the *science of solving equations*.

Algebra

Most of the major ancient civilizations, the Babylonian, Egyptian, Chinese, and Hindu, dealt with the solution of polynomial equations, mainly linear and quadratic equations. The Babylonians (c. 1700 BC) were particularly proficient algebraists. They were able to solve quadratic equations by methods similar to ours.

Linear algebra

Among the elementary concepts of linear algebra are:

- linear equations,
- matrices,
- determinants,
- linear transformations,
- linear independence,
- dimension,
- vector spaces,
- and many others.

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- and many others.

All these concepts are closely interconnected. For instance, it is difficult to disengage matrices and systems of linear equations.

An application

Trail mix packaging

Suppose you are the production manager at a food-packaging plant and one of your product lines is trail mix, a healthy snack popular with hikers and backpackers, containing raisins, peanuts and hard-shelled chocolate pieces. By adjusting the mix of these three ingredients, you are able to sell three varieties of this item. The fancy version is sold in half-kilogram packages at outdoor supply stores and has more chocolate and fewer raisins, thus commanding a higher price. The standard version is sold in one kilogram packages in grocery stores and gas station mini-markets. Since the standard version has roughly equal amounts of each ingredient, it is not as expensive as the fancy version. Finally, a bulk version is sold in bins at grocery stores for consumers to load into plastic bags in amounts of their choosing. To appeal to the shoppers that like bulk items for their economy and healthfulness, this mix has many more raisins (at the expense of chocolate) and therefore sells for less.

Trail mix packaging

Your production facilities have limited storage space and early each morning you are able to receive and store 380 kilograms of raisins, 500 kilograms of peanuts and 620 kilograms of chocolate pieces. As production manager, one of your most important duties is to decide **how much of each version of trail mix to make every day**. Clearly, you can have up to 1500 kilograms of raw ingredients available each day, so to be the most productive you will likely produce 1500 kilograms of trail mix each day. Also, you would prefer not to have any ingredients leftover each day, so that your final product is as fresh as possible and so that you can receive the maximum delivery the next morning. But how should these ingredients be allocated to the mixing of the bulk, standard and fancy versions? First, we need a little more information about the mixes. Workers mix the ingredients in 15 kilogram batches, and each row of the table below gives a recipe for a 15 kilogram batch. There is some additional information on the costs of the ingredients and the price the manufacturer can charge for the different versions of the trail mix.

Trail mix packaging

	Raisins (kg/batch)	Peanuts (kg/batch)	Chocolate (kg/batch)	Cost ($/kg)	Sale price ($/kg)
Bulk	7	6	2	3.69	4.99
Standard	6	4	5	3.86	5.50
Fancy	2	5	8	4.45	6.50
Storage (kg)	380	500	620		
Cost ($/kg)	2.55	4.65	4.80		

Trail mix packaging

Let us denote the amount of each mix to produce each day, measured in kilograms, by the variable quantities b,s and f. Your production schedule can be described as values of b, s and f that do several things. First, we cannot make negative quantities of each mix, so

$$b \geq 0 \qquad s \geq 0 \qquad f \geq 0$$

Trail mix packaging

Second, if we want to consume all of our ingredients each day, the storage capacities lead to three linear equations, one for each ingredient,

$$\frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f = 380$$

$$\frac{6}{15}b + \frac{4}{15}s + \frac{5}{15}f = 500$$

$$\frac{2}{15}b + \frac{5}{15}s + \frac{8}{15}f = 620$$

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As production manager, you solve the system of equations and find it has a unique solution:

$$b = 300 \text{ kg} \qquad s = 300 \text{ kg} \qquad f = 900 \text{ kg}$$

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Although not asked for, you can compute the profit derived from your decision:

$$300(4.99 - 3.69) + 300(5.50 - 3.86) + 900(6.50 - 4.45) = 2727.00$$

Trail mix packaging

Some months later, the marketing department adjusts the recipes and asks you to solve the same problem with the new data.

	Raisins (kg/batch)	Peanuts (kg/batch)	Chocolate (kg/batch)	Cost ($/kg)	Sale price ($/kg)
Bulk	7	**5**	**3**	**3.70**	4.99
Standard	6	**5**	**4**	**3.85**	5.50
Fancy	2	5	8	4.45	6.50
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$$\frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f = 380$$
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$$\frac{5}{15}b + \frac{5}{15}s + \frac{5}{15}f = 500$$
$$\frac{3}{15}b + \frac{4}{15}s + \frac{8}{15}f = 620$$

$$b = 4f - 3300$$
$$s = -5f + 4800$$

Let f be free.

Trail mix packaging

$$\frac{7}{15}(4f-3300)+\frac{6}{15}(-5f+4800)+\frac{2}{15}f=0f+\frac{5700}{15}=380$$
$$\frac{5}{15}(4f-3300)+\frac{5}{15}(-5f+4800)+\frac{5}{15}f=0f+\frac{7500}{15}=500$$
$$\frac{3}{15}(4f-3300)+\frac{4}{15}(-5f+4800)+\frac{8}{15}f=0f+\frac{9300}{15}=620$$

Trail mix packaging

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However, from a practical point of view, the system has a finite number of solutions because f is most likely an integer (we cannot sell fractional bags of fancy mix).

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Besides, remember that we cannot make negative amounts of each mix. Therefore,

$$b \geq 0 \quad \rightarrow \quad 4f - 3300 \geq 0 \quad \rightarrow \quad f \geq 825$$

$$s \geq 0 \quad \rightarrow \quad -5f + 4800 \geq 0 \quad \rightarrow \quad f \leq 960$$

Trail mix packaging

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However, from a practical point of view, the system has a finite number of solutions because f is most likely an integer (we cannot sell fractional bags of fancy mix).

Besides, remember that we cannot make negative amounts of each mix. Therefore,

$$\begin{array}{ccccc} b \geq 0 & \rightarrow & 4f - 3300 \geq 0 & \rightarrow & f \geq 825 \\ s \geq 0 & \rightarrow & -5f + 4800 \geq 0 & \rightarrow & f \leq 960 \end{array}$$

So, you really have to choose the "best" value for f from the following finite set:

$$\{825, 826, ..., 960\}$$

Trail mix packaging

What would be a reasonable objective (function) to optimize?

Trail mix packaging

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$$\begin{aligned}\text{Profit} &= (4f - 3300)(4.99 - 3.70) + (-5f + 4800)(5.50 - 3.85) + (f)(6.50 - 4.45) \\ &= -1.04f + 3663\end{aligned}$$

Trail mix packaging

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So, what is the best decision?

$$f = 825 \quad \rightarrow \quad b = 4(825) - 3300 = 0 \quad \rightarrow \quad s = -5(825) + 4800 = 675$$

Trail mix packaging

Some time later, the sales department decided to reduce the price of the standard mix to 5.25 ($/kg). So, you adjust the profit (objective) function:

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$$\begin{aligned}\text{Profit} &= (4f - 3300)(4.99 - 3.70) + (-5f + 4800)(\mathbf{5.25} - 3.85) + (f)(6.50 - 4.45)\\ &= 0.21f + 2463\end{aligned}$$

Under this scenario, what is the best decision?

Trail mix packaging

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Under this scenario, what is the best decision?

$$f = 960 \quad \rightarrow \quad b = 4(960) - 3300 = 540 \quad \rightarrow \quad s = -5(960) + 4800 = 0$$

Ending

Summary

- Systems of linear equations help us model problems.
- Systems of linear equations can have one solution or infinite solutions.
- Sometimes, the system has no solution.
- We will learn how to identify each case.

Homework

- Reproduce the trail mix packaging problem using LaTeX. Besides, solve the next bullet.
- In our trail mix packaging example two different prices were considered for marketing standard mix with the revised recipes (one-third peanuts in each recipe). Selling standard mix at 5.50 resulted in selling the minimum amount of the fancy mix and no bulk mix. At 5.25 it was best for profits to sell the maximum amount of fancy mix and then sell no standard mix. Determine a selling price for standard mix that allows for maximum profits while still selling some of each type of mix.

Next topics

- Systems of linear equations
- Augmented matrix
- Reduced row-echelon form.
- Matrices.

Thank you