Linear algebra

Systems of linear equations - 1 of 2

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CONAHCYT
INAOE

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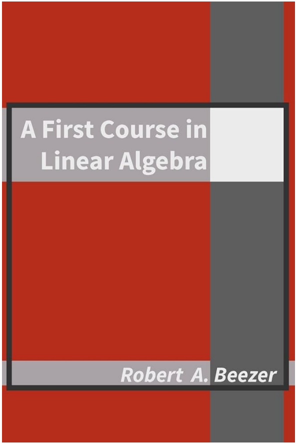
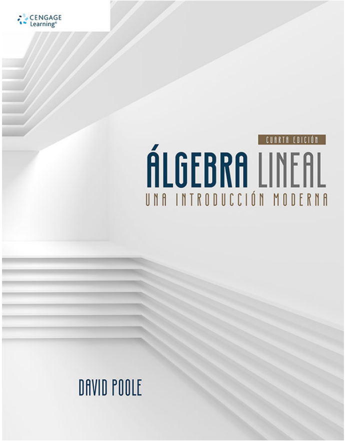
Contents

Bibliography





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Systems of linear equations

Solving systems of linear equations

In this context, "to solve" a system of equations means to find all of the values of some variables quantities that make an equation, or several equations, true.

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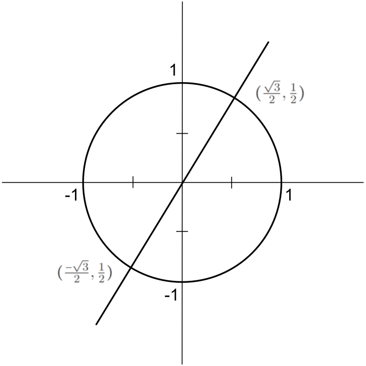

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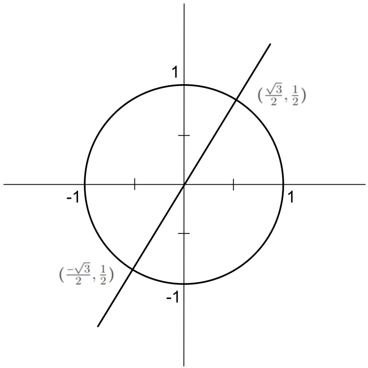

$$S = \left\{ \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \right\}$$

Systems of linear equations

A system of linear equations is a collection of m equations in the variable quantities x_1,x_2,x_3,...,x_n of the form,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where the value of a_{ij}, b_i and x_j, $1 \leq i \leq m$, $1 \leq j \leq n$, are from the set of complex numbers, $\mathbb{C}$.

Solution of a system of linear equations

A solution of a system of linear equations in n variables, x_1,x_2,x_3,...,x_n, is an ordered list of n complex numbers, s_1,s_2,s_3,...,s_n such that if we substitute s_1 for x_1, s_2 for x_2, s_3 for x_3, ..., s_n for x_n, then for every equation of the system the left side will equal the right side, i.e., each equation is true simultaneously.

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More typically, we will write a solution in a form like $x_1 = 12$, $x_2 = -7$, $x_3 = 2$ to mean that $s_1 = 12$, $s_2 = -7$, $s_3 = 2$.

Possibilities for solution sets

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 - x_2 &= 2 \end{aligned}$$

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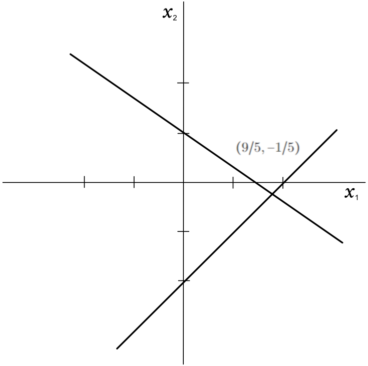

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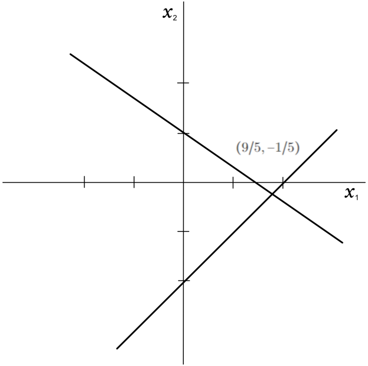

One solution: $x_1 = \frac{9}{5}$ and $x_2 = -\frac{1}{5}$

Possibilities for solution sets

$$
\begin{aligned}
2x_1 + 3x_2 &= 3 \\
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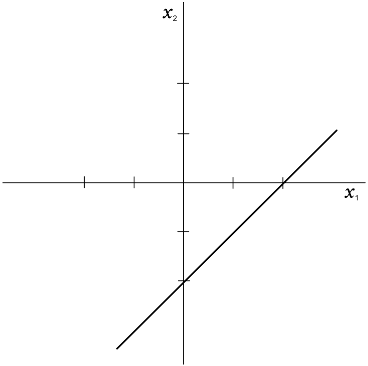

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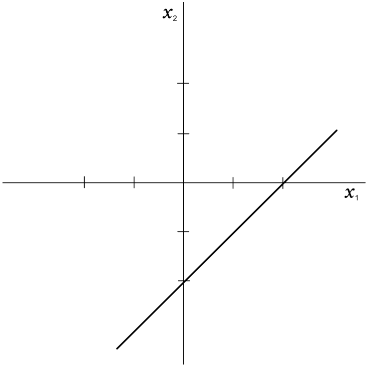

Infinite solutions (shortly, we will see how to describe this infinite set)

Possibilities for solution sets

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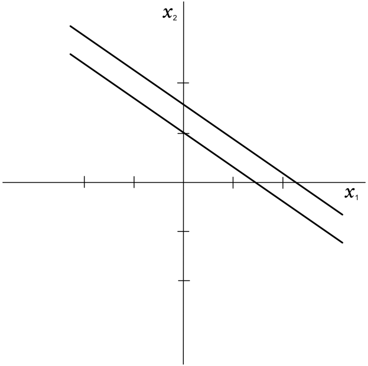

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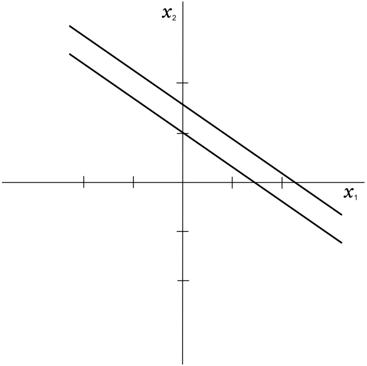

An empty solution set, $S = \emptyset$

Equivalent systems

Two systems of linear equations are equivalent if their solution sets are equal.

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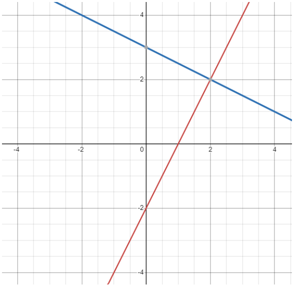
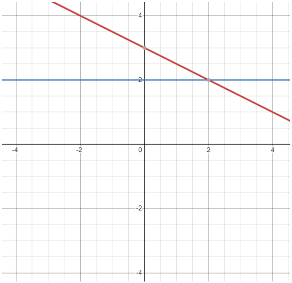
Two systems of linear equations are equivalent if their solution sets are equal.

$$\begin{aligned} 2x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 6 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ x_2 &= 2 \end{aligned}$$





Equation operations

Given a system of linear equations, the following three operations will transform the given system into a different one, and each operation is known as an equation operation.

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero quantity.
3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

Equation operations preserve solution sets

The first equation operation, swapping locations of equations, evidently preserve solution sets.

The next equation operations are less obvious. Let us prove the second one: multiply each term of an equation by a nonzero quantity.

Proving the second equation operation

Suppose $\alpha \neq 0$ is a number. Let's choose to multiply the terms of equation i by α to build the new system of equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
\alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \alpha a_{i3}x_3 + \cdots + \alpha a_{in}x_n &= \alpha b_i \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
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a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Let S denote the solution set to the original system, and let T denote the solution set of the transformed system. How do we prove two sets are equal ($S = T$)?

(a) Show $S \subseteq T$

Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$ is a solution to the original system. Ignoring the i-th equation for a moment, we know it makes all the other equations of the transformed system true. We also know that

$$a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i$$

which we can multiply by α to get

$$\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \alpha a_{i3}\beta_3 + \cdots + \alpha a_{in}\beta_n = \alpha b_i$$

Thus, the i-th equation of the transformed system is also true. Therefore, $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$ and $S \subseteq T$.

(b) Show $T \subseteq S$

Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$ is a solution to the transformed system. Ignoring the i-th equation for a moment, we know it makes all the other equations of the original system true. We also know that

$$\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \alpha a_{i3}\beta_3 + \cdots + \alpha a_{in}\beta_n = \alpha b_i$$

which we can multiply by $\frac{1}{\alpha}$, since $\alpha \neq 0$, we get

$$a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i$$

Thus, the i-th equation of the original system is also true. So, $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$, and therefore $T \subseteq S$. By (a) and (b), $S = T$.

Proving the third equation operation

This is left as homework.

Reduced row-echelon form

Matrix

After solving a few systems of equations, you will recognize that it doesn't matter so much what we call our variables, as opposed to what number act as their coefficients. So, we can isolate the important bits a system of linear equations using a matrix.

An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A,B,C,...) to denote matrices and squared-off bracket to delimit the layout. Rows are numbered from top to bottom and columns from left to right, always starting from 1. The notation $[A]_{ij}$ refers to the complex number in row i and column j of A.

Column vector

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as **u**, **v**, **w**, **x**, **y**, **z**. To refer to the **entry** or **component** of vector **v** in location i of the list, we write $[\mathbf{v}]_i$.

Zero column vector

The **zero vector** of size m is the column vector of size m where each component is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $[\mathbf{0}]_i = 0$ for $1 \leq i \leq m$.

Coefficient matrix

For a system of linear equations,

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\end{aligned}
$$

the **coefficient matrix** is the $m \times n$ matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}
$$

Vector of constants

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solution vector

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The solution vector may do double-duty on occasion (sometimes refers to variables and some times to a solution).

Matrix representation of a linear system

If A is the coefficient matrix of a system of linear equations and $\mathbf{b}$ is the vector of constant, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

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If A is the coefficient matrix of a system of linear equations and **b** is the vector of constant, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

$$\mathcal{LS}(A, \mathbf{b})$$

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 + 5x_4 &= 9 \\ 3x_1 + x_2 + x_4 &= 0 \\ -2x_1 + 7x_2 - 5x_3 + 2x_4 &= -3 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 4 & -3 & 5 \\ 3 & 1 & 0 & 4 \\ -2 & 7 & -5 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}$$

Augmented matrix

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants $\mathbf{b}$. Then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column $(n+1)$ is the column vector $\mathbf{b}$. This matrix will be represented as $[A \mid \mathbf{b}]$.

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$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

Row operations

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. leave the first row the same after this operation, but replace the second row by the new values.

Row operations

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
2. αR_i : Multiply row i by the nonzero scalar α.
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

Row-equivalent matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

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Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 & 6 \\ 5 & 2 & -2 & 3 \\ 2 & -1 & 3 & 4 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

Row-equivalent matrices represent equivalent systems

Theorem

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

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Each row operation has the same effect of each equation operation. So, the solution set for the corresponding system of linear equations is preserved. □

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So our strategy is to begin with a system of equations, represent it by an augmented matrix, perform row operations to get a "simpler" augmented matrix, convert back to a "simpler" system of linear equations and then solve that system, knowing that its solutions are those of the original system.

Example

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 1 \\ x_3 &= 4 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-1R_1 + R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{-2R_2 + R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} = B$$

Reduced row-echelon form

A matrix is in **reduced row-echelon form** if it meets all the following conditions:

1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $s > i$, then $t > j$.

Reduced row-echelon form

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called **leading 1**. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, ..., d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, ..., f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

Reduced row-echelon form

$$\begin{bmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Existence and uniqueness

Theorem

Suppose A is a matrix. Then there is a matrix B so that

1. *A and B are row-equivalent.*
2. *B is in reduced row-echelon form.*

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Proof.

The proof is by construction. The procedure that returns B, given an input A, is known as **Gauss-Jordan elimination**. □

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Theorem

The reduced row-echelon form of a given matrix is unique (we omit the proof).

Gauss-Jordan elimination

1. Set $j = 0$ and $r = 0$.
2. Increase j by 1. If j now equals $n+1$, then stop.
3. Examine the entries of A in column j located in rows $r+1$ through m. If all of these entries are zero, then go to Step 2.
4. Choose a row from rows $r+1$ through m with a nonzero entry in column j. Let i denote the index for this row.
5. Increase r by 1.
6. Use the first row operation to swap rows i and r.
7. Use the second row operation to convert the entry in row r and column j to a 1.
8. Use the third row operation with row r to convert every other entry of column j to zero.
9. Go to Step 2.

Gauss-Jordan elimination

```
input m, n and A
r := 0
for j := 1 to n
   i := r+1
   while i <= m and A[i,j] == 0
       i := i+1
   if i < m+1
       r := r+1
       swap rows i and r of A (row op 1)
       scale A[r,j] to a leading 1 (row op 2)
       for k := 1 to m, k <> r
           make A[k,j] zero (row op 3, employing row r)
output r and A
```

The intuitive mechanism

- Gauss-Jordan elimination explores the matrix column by column, from left to right.
- r keeps track of the number of nonzero rows (equivalently, the number of 1-pivots).
- At every "round", the three row operations are performed:
  1. $R_i \leftrightarrow R_r$ (if convenient).
  2. αR_r (to create a leading 1).
  3. $\alpha R_r + R_k$ (to turn into 0 all the other entries in the column).
- Notice that once an entry is turned into 0, future row operations cannot change that value. Thus, at the end, our matrix has the desired row-echelon form.

One solution

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned} \qquad \begin{aligned} x_1 &= -3 \\ x_2 &= 5 \\ x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{bmatrix} \xrightarrow{input} \boxed{Gauss - Jordan} \xrightarrow{output} \begin{bmatrix} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

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$$S = \left\{ \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \right\}$$

Infinite solutions

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned} \qquad \begin{aligned} x_1 + x_3 &= 3 \\ x_2 - x_3 &= 2 \\ 0x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{bmatrix} \xrightarrow{input} \boxed{Gauss - Jordan} \xrightarrow{output} \begin{bmatrix} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$S = \left\{ \begin{bmatrix} 3 - x_3 \\ 2 + x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{C} \right\}$$

Zero solutions

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 0 \\ x_2 + x_3 - 3x_4 &= 0 \\ 0 &= 1 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{bmatrix} \xrightarrow{input} \boxed{Gauss - Jordan} \xrightarrow{output} \begin{bmatrix} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Zero solutions

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned} \qquad \begin{aligned} x_1 + 3x_3 - 2x_4 &= 0 \\ x_2 + x_3 - 3x_4 &= 0 \\ 0 &= 1 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{bmatrix} \xrightarrow{input} \boxed{Gauss - Jordan} \xrightarrow{output} \begin{bmatrix} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = \emptyset$$

Ending

Summary

- Systems of linear equations may have one solution, infinite solutions, or zero solutions.

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- Systems of linear equations may have one solution, infinite solutions, or zero solutions.
- The important bits of a system of linear equations lie in its augmented matrix.
- Equation operations are equivalent to row operations.
- Every matrix has a unique row-equivalent matrix in reduced row-echelon form. Gauss-Jordan elimination let us find it.

Homework

- Find all the six-digit numbers in which the first digit is one less than the second, the third digit is half the second, the fourth digit is three times the third, the last two digits sum a number that equals the sum of the fourth and fifth, and the fourth and sixth digits are equal. The sum of all the digits is 24.

Next topics

- Consistent (and inconsistent) systems.
- Homogeneous systems.
- Nonsingular matrices.

Thank you