Linear algebra

Systems of linear equations - 2 of 2

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CONAHCYT
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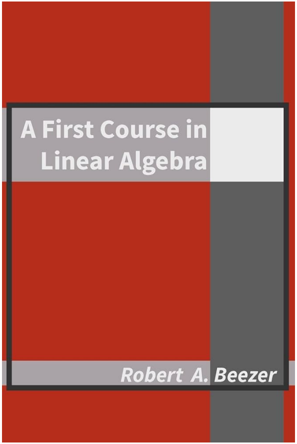
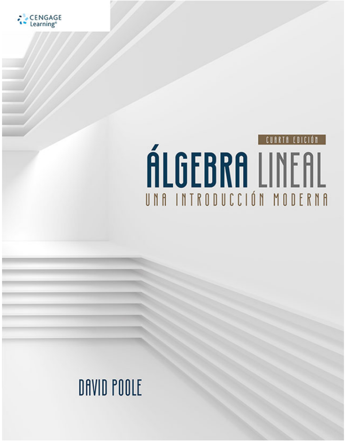
Contents

Bibliography





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Consistent systems

Consistent systems

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A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

So, given a system of linear equations, we would like to know if it is consistent (or inconsistent) by analyzing its reduced row-echelon form.

Notation

Let us use D and F to denote the indices of the columns. Besides, we will use r to denote the number of nonzero rows. For example,

$$B = \begin{bmatrix} \boxed{1} & 5 & 0 & 0 & 2 & 8 & 0 & 5 & -1 \\ 0 & 0 & \boxed{1} & 0 & 4 & 7 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 9 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r = 4 \qquad d_1 = 1 \qquad d_2 = 3 \qquad d_3 = 4 \qquad d_4 = 7$$

$$f_1 = 2 \qquad f_2 = 5 \qquad f_3 = 6 \qquad f_4 = 8 \qquad f_5 = 9$$

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$$r = 4 \qquad d_1 = 1 \qquad d_2 = 3 \qquad d_3 = 4 \qquad d_4 = 7$$

$$f_1 = 2 \qquad f_2 = 5 \qquad f_3 = 6 \qquad f_4 = 8 \qquad f_5 = 9$$

$D = \{d_1, d_2, d_3, d_4\} = \{1, 3, 4, 7\}$ and $F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 8, 9\}$ define a **partition** of the columns in B.

The number r

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- the number of nonzero rows,
- the number of leading 1's,
- and the number of pivot columns.

An example

$$
\begin{aligned}
x_1 - x_2 - 2x_3 + x_4 + 11x_5 &= 13 \\
x_1 - x_2 + x_3 + x_4 + 5x_5 &= 16 \\
2x_1 - 2x_2 + x_4 + 10x_5 &= 21 \\
2x_1 - 2x_2 - x_3 + 3x_4 + 20x_5 &= 38 \\
2x_1 - 2x_2 + x_3 + x_4 + 8x_5 &= 22
\end{aligned}
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$D = \{1, 3, 4\}$. Thus, x_1, x_3, and x_4 are **dependent variables**. There are $r = |D| = 3$ equations. $F = \{2, 5, 6\}$. Thus x_2 and x_5 are **free variables** ($6 \in F$ corresponds to the vector of constants).

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$$\begin{aligned} &(x_{d_1} = x_1) \quad x_1 = 6 + x_2 - 3x_5 \\ &(x_{d_2} = x_3) \quad x_3 = 1 + 2x_5 \\ &(x_{d_3} = x_4) \quad x_4 = 9 - 4x_5 \end{aligned}$$

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&(x_{d_3} = x_4) \quad x_4 = 9 - 4x_5
\end{aligned}
\qquad
S = \left\{ \begin{bmatrix} 6 + x_2 - 3x_5 \\ x_2 \\ 1 + 2x_5 \\ 9 - 4x_5 \\ x_5 \end{bmatrix} : x_2, x_5 \in \mathbb{C} \right\}
$$

Recognizing consistency of a linear system

Theorem

Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n+1$ of B.

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Proof.

($\Leftarrow$) Leading 1 of row r is in column $n+1$. Thus, $0=1$. The system is inconsistent.
($\Rightarrow$) We prove the contrapositive: If the leading 1 or row r is not in column $n+1$, then the system of equations is consistent. Since the last leading 1 is not in the last column, no leading 1 for any row is in the last column (because of the properties of a reduced row-echelon form). We can construct a solution by setting

$$x_{d_i} = [B]_{i,n+1} \ , \ 1 \le i \le r \qquad\qquad x_{f_i} = 0 \ , \ 1 \le i \le n-r$$

□

Recognizing consistency of a linear system

The beauty of this theorem is that we can unequivocally test if a system is consistent or inconsistent by looking at just a single entry. Also, notice that

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- for a consistent system, the row-reduced augmented matrix has $n+1 \in F$,

Recognizing consistency of a linear system

The beauty of this theorem is that we can unequivocally test if a system is consistent or inconsistent by looking at just a single entry. Also, notice that

- for a consistent system, the row-reduced augmented matrix has $n + 1 \in F$,
- for an inconsistent system, $n + 1 \in D$. In this scenario, it makes no sense to label variables as dependent or free because there are no solutions.

Inconsistent system, r and n

Theorem

Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Inconsistent system, r and n

Theorem

Suppose A is the augmented matrix of a system of linear equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Proof.

If $r = n + 1$, then $D = \{1, 2, 3, ..., n, n + 1\}$ and every column of B contains a leading 1 and is a pivot column. By the previous theorem, the system is inconsistent. □

Consistent systems, r and n

Theorem

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Consistent systems, r and n

Theorem

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Proof.

Since B has $n + 1$ columns, $r \leq n + 1$. However, we know the system is consistent; therefore, by the previous theorem, $r \leq n$. We now prove two implications.

(1) When $r = n$, we find $n - r = 0$ free variables (i.e. $F = \{n + 1\}$) and there is a unique solution.

(2) When $r < n$, we have $n - r > 0$ free variables. Thus, there are infinite solutions. □

Free variables

Theorem

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Free variables

Theorem

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Proof.

See the previous theorem. □

Possible solution sets for linear systems

Theorem

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Possible solution sets for linear systems

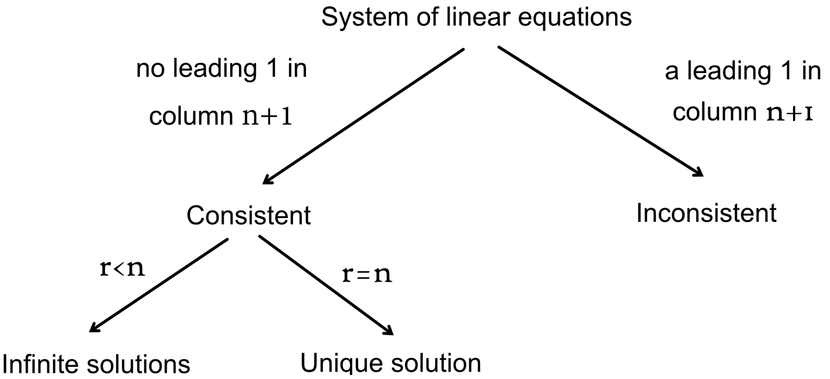
Theorem

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Proof.

By its definitions, a system is either inconsistent or consistent. The first case describes systems with no solutions. For consistent systems, we have two possibilities guaranteed by one of the previous theorems: one solution or infinite solutions. □

Possible solution sets for linear systems



Consistent, more variables than equations, infinite solutions

Theorem

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Consistent, more variables than equations, infinite solutions

Theorem

Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Proof.

Suppose that the augmented matrix of the system of equations is row-equivalent to B, a matrix in reduced row-echelon form with r nonzero rows. Because B has m rows in total, $r \leq m$. Follow this with the hypothesis that $n > m$ and we find that the system has a solution set described by at least one free variable because

$$n - r \geq n - m > 0$$

A consistent system with free variables has an infinite number of solutions. □

Homogeneous systems of equations

Homogeneous system

Definition

A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, if $\mathbf{b} = \mathbf{0}$

Homogeneous system

Theorem

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Homogeneous system

Theorem

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Proof.

Set each variable of the system to zero. When substituting these values into each equation, the left-hand side evaluates to zero, no matter what the coefficients are. Since a homogeneous system has zero on the right-hand side of each equation as the constant term, each equation is true. With one demonstrated solution, we can call the system consistent. □

Homogeneous system

Definition

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0$,..., $x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the **trivial solution**.

Homogeneous, more variables than equations, infinite solutions

Theorem

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Homogeneous, more variables than equations, infinite solutions

Theorem

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Proof.

We are assuming the system is homogeneous, so it is consistent. We know that consistent systems with $n > m$ have infinitely many solutions. □

Null space of a matrix

Definition

The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

Computing a null space, no. 1

Let's compute the null space $\mathcal{N}(A)$ of

$$A = \begin{bmatrix} 2 & -1 & 7 & -3 & -8 \\ 1 & 0 & 2 & 4 & 9 \\ 2 & 2 & -2 & -1 & 8 \end{bmatrix}$$

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We desire to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. So, we row-reduce the augmented matrix to obtain

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We desire to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. So, we row-reduce the augmented matrix to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 1 & 0 \\ 0 & \boxed{1} & -3 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 1 & 0 \\ 0 & \boxed{1} & -3 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 \end{bmatrix}$$

x_3 and x_5 are free variables. So,

$$\begin{aligned} x_1 &= -2x_3 - x_5 \\ x_2 &= 3x_3 - 4x_5 \\ x_4 &= -2x_5 \end{aligned}$$

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x_3 and x_5 are free variables. So,

$$\begin{aligned} x_1 &= -2x_3 - x_5 \\ x_2 &= 3x_3 - 4x_5 \\ x_4 &= -2x_5 \end{aligned}$$

Thus, the null space (solution set) is

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 - x_5 \\ 3x_3 - 4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} : x_3, x_5 \in \mathbb{C} \right\}$$

Computing a null space, no. 2

Let's compute the null space $\mathcal{N}(C)$ of

$$C = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}$$

Computing a null space, no. 2

Let's compute the null space $\mathcal{N}(C)$ of

$$C = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}$$

We desire to solve the homogeneous system $\mathcal{LS}(C, \mathbf{0})$. So, we row-reduce the augmented matrix to obtain

Computing a null space, no. 2

Let's compute the null space $\mathcal{N}(C)$ of

$$C = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}$$

We desire to solve the homogeneous system $\mathcal{LS}(C, \mathbf{0})$. So, we row-reduce the augmented matrix to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Let's compute the null space $\mathcal{N}(C)$ of

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We desire to solve the homogeneous system $\mathcal{LS}(C, \mathbf{0})$. So, we row-reduce the augmented matrix to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables ($F = \{4\}$). So,

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 0$$

Computing a null space, no. 2

Let's compute the null space $\mathcal{N}(C)$ of

$$C = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}$$

We desire to solve the homogeneous system $\mathcal{LS}(C, \mathbf{0})$. So, we row-reduce the augmented matrix to obtain

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$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Thus, the null space (solution set) is

$$\mathcal{N}(C) = \{\mathbf{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Nonsingular matrices

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This section focuses on matrices with equal number of rows and columns, which when considered as coefficient matrices lead to systems with equal number of equations and variables.

Nonsingular matrices

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Observation

- *A system of equations is not a matrix.*
- *A matrix is not a system of equations.*
- *A matrix is not a solution set.*
- *A solution set is not a system of equations.*

Square matrix

Definition

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize when a matrix is not square, we will call it **rectangular**.

Nonsingular matrix

Definition

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, in other words, the system has only the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise, we say A is a **singular** matrix.

Nonsingular matrix

Definition

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, in other words, the system has only the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise, we say A is a **singular** matrix.

Observation

- *We can decide nonsingularity for any square matrix.*
- *The determination of nonsingularity involves the solution set for a certain homogeneous system of equations.*
- *It makes no sense to call a system of equations nonsingular.*
- *It makes no sense to call a* 5×7 *matrix singular.*

Identity matrix

Definition

The $m \times m$ **identity matrix**, I_m, is defined by

$$[I_m]_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad 1 \leq i, j \leq m$$

Identity matrix

Definition

The $m \times m$ **identity matrix**, I_m, is defined by

$$[I_m]_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad 1 \leq i, j \leq m$$

Notice that an identity matrix is square and in reduced row-echelon form.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Nonsingular matrices row-reduce to the identity matrix

Theorem

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Nonsingular matrices row-reduce to the identity matrix

Theorem

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Proof.

($\Leftarrow$) If B is the identity matrix, $n = r$, i.e., there are no free variables. Therefore, $\mathcal{LS}(A, \mathbf{0})$ has only the trivial solution. This is the definition of a nonsingular matrix.
($\Rightarrow$) If A is nonsingular, $\mathcal{LS}(A, \mathbf{0})$ only has the trivial solution. Thus, it has $n - r = 0$ free variables. Therefore, $n = r$, which means that B has n pivot columns. Since B is a square matrix, it must be the identity matrix. □

Nonsingular matrices have trivial null spaces

Theorem

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.

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Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A, $\mathcal{N}(A)$, contains only the zero vector, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.

Proof.

It follows from the definitions. □

Nonsingular matrices and unique solutions

Theorem

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Nonsingular matrices and unique solutions

Theorem

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Proof.

($\Leftarrow$) Assume that $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of $\mathbf{b}$. So, let's choose $\mathbf{b} = \mathbf{0}$. As we already know, $\mathcal{LS}(A, \mathbf{0})$ has only the trivial solution. But this is precisely the definition of a nonsingular matrix.
($\Rightarrow$) Assuming A is nonsingular of size $n \times n$, its reduced row-echelon form is the identity matrix. Form the augmented matrix $A' = [A|\mathbf{b}]$ and apply the same sequence of row operations. The result is the matrix $B' = [I_n|\mathbf{c}]$, which is in reduced row-echelon with $r = n$. So, vector $\mathbf{c}$ is the only solution because there are no free variables $(n - r = n - n = 0)$. □

Nonsingular matrix equivalences

Theorem

Suppose that A is a square matrix. The following are equivalent.

- A *is nonsingular.*
- A *row-reduces to the identity matrix.*
- *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
- *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*

Nonsingular matrix equivalences

Theorem

Suppose that A is a square matrix. The following are equivalent.

- *A is nonsingular.*
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- *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
- *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*

Proof.

It follows from the previous theorems and definitions. □

Ending

Summary

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- r let us identify consistency of a system (and type of consistency).
- A **homogeneous** system is always consistent (it has the **trivial solution**).
- The **null space**, $\mathcal{N}(A)$, of a matrix A is the set of all vectors that are solutions to $\mathcal{LS}(A, \mathbf{0})$.

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- Matrix A is **nonsingular** if $\mathcal{N}(A) = \mathbf{0}$.
- Applying Gauss-Jordan elimination to a nonsingular matrix always leads to the **identity matrix**.
- Systems $\mathcal{LS}(A, \mathbf{b})$ have only one solution, for any $\mathbf{b}$, if and only if A is nonsingular.

Homework

- Suppose that the coefficient matrix of a consistent system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions.
- Consider the homogeneous system of linear equations $\mathcal{LS}(A, \mathbf{0})$, and suppose that $\mathbf{u} = [u_1, u_2, ..., u_n]$ is one solution to the system. Prove that $\mathbf{v} = [4u_1, 4u_2, ..., 4u_n]$ is also a solution to $\mathcal{LS}(A, \mathbf{0})$.

Homework

- If any, find the values of k for which each system has (a) no solution, (b) one solution, (c) infinite solutions. (Use Gauss-Jordan elimination; avoid using the determinant concept.)

$$\begin{aligned} kx + 2y &= 3 \\ 2x - 4y &= -6 \end{aligned} \qquad \begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$$

Next topics

Vectors

Thank you