# Linear algebra

## Vectors

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INAOE

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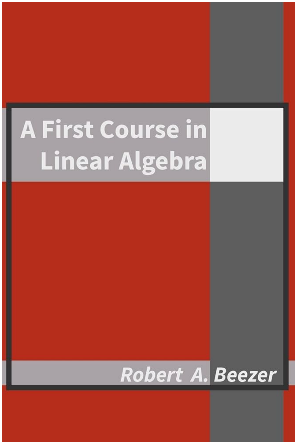
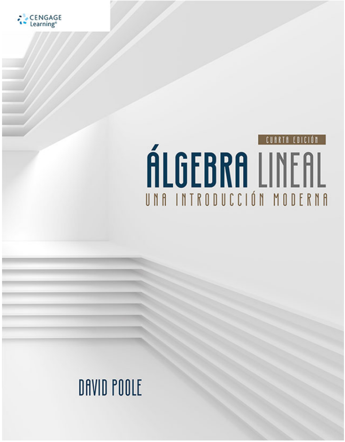
# Contents

# Bibliography

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- Usually represented by arrows that have:
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Physics

- Usually represented by arrows that have:
  - magnitude
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Mathematics

- Anything.
- As long as it respects certain rules.

Computer science

$$= \begin{bmatrix} 200{,}000 \\ 4 \\ 60 \end{bmatrix}$$

- List of numbers.

# Vectors in the plane

**Definition**

A **vector** is a directed line segment that corresponds to a displacement from one point $A$ to another point $B$. The vector from $A$ to $B$ is denoted by $\overrightarrow{AB}$; the point $A$ is called its **initial point**, or **tail**, and the point $B$ is called its **terminal point** or **head**. Often, a vector is simply denoted by a single boldface, lowercase letters such as **v**.

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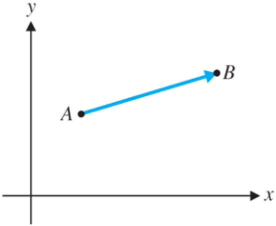


# Vectors in the plane

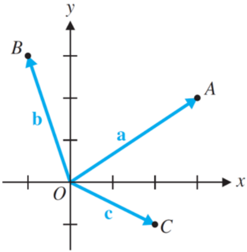
The set of all points in the plane corresponds to the set of all vector whose tail are at the origin $O$.

## Definition

Vectors with its tail at the origin are called **position vectors**.

# Vectors in the plane

Point $A$ corresponds to the position vector $\mathbf{a} = \overrightarrow{OA} = [3, 2]$. The other vectors in the figure are $\mathbf{b} = [-1, 3]$ and $\mathbf{c} = [2, -1]$.



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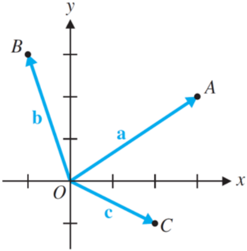



The individual coordinates (3 and 2 in the case of **a**) are called the **components** of the vector.

# Vectors in the plane

Two vectors are equal if and only if their corresponding components are equal. Thus, $[x, y] = [1, 5]$ implies that $x = 1$ and $y = 5$.

Using **column vectors** instead of **row vectors** is frequently convenient.

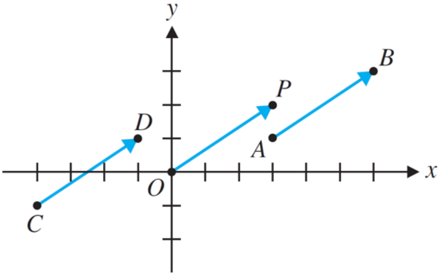
So, $[3, 2]$ can be represented as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

# Vectors in the plane

We cannot draw the vector $[0, 0] = \overrightarrow{OO}$ from the origin to itself. Nevertheless, it is a perfectly good vector and has a special name: the **zero vector**. The zero vector is denoted by $\mathbf{0}$.

# Vectors in the plane

What can you say about these three vectors?



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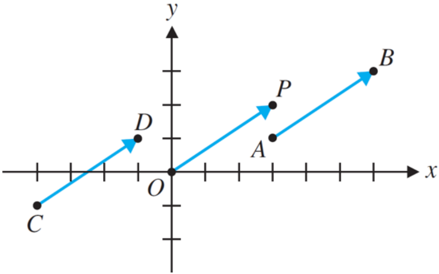
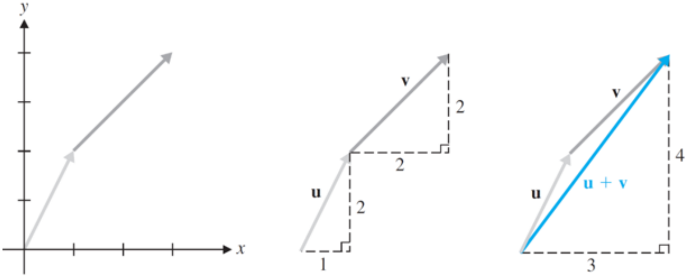



By setting the tail of each vector in the origin, we observe they all coincide.

# New vectors from old

We often want to follow one vector by another. This leads to the notion of **vector addition**.

If we follow **u** by **v**, we can visualize the total displacement as a third vector, denoted by $\mathbf{u} + \mathbf{v}$.



# New vectors from old

In general, if $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$, the their **sum** $\mathbf{u} + \mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$$

# New vectors from old

Our next vector operation is **scalar multiplication**. Given a vector **v** and a real number $c$, the **scalar multiplication** $c\mathbf{v}$ is the vector contained by multiplying each component of **v** by $c$. In general,

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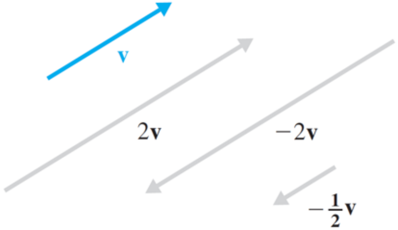
Geometrically, $c\mathbf{v}$ is a "scaled" version of **v**.



# Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is a shorthand for $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, the cartesian product of $\mathbb{R}$ with itself $n$ times. So, it is the set of all ordered $n$-tuples of real numbers written as row or column vectors. Thus, a vector $\mathbf{v} \in \mathbb{R}^n$ is of the form

$$[v_1, v_2, .., v_n] \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The individual entries of $\mathbf{v}$ are its components; $v_i$ is called the $i$-th component.

# Vectors in $\mathbb{R}^n$

We extend the definitions of vector addition and scalar multiplication to $\mathbb{R}^n$ in the obvious way:

If $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., v_n]$, the $i$-th component of $\mathbf{u} + \mathbf{v}$ is $u_i + v_i$ and the $i$-th component of c**v** is just $cv_i$.

# Vectors in $\mathbb{R}^n$

Algebraic properties of vectors in $\mathbb{R}^n$.

**Theorem**

*Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* $c$ *and* $d$ *be scalars. Then*

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- $1\mathbf{u} = \mathbf{u}$ *(one)*

# Vectors in $\mathbb{R}^n$

Each bullet must be proved. In general, they all inherit the properties of the operations over real numbers. For instance,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}=\begin{bmatrix}v_1+u_1\\v_2+u_2\\\vdots\\v_n+u_n\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}+\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\mathbf{v}+\mathbf{u}$$

# Example

Simplify ($\mathbf{x}$ in terms of $\mathbf{a}$)

$$5\mathbf{x} - \mathbf{a} = 2(\mathbf{a} + 2\mathbf{x})$$

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Simplify (**x** in terms of **a**)

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5\mathbf{x} - \mathbf{a} &= 2\mathbf{a} + 4\mathbf{x} \\
(5\mathbf{x} - \mathbf{a}) - 4\mathbf{x} &= (2\mathbf{a} + 4\mathbf{x}) - 4\mathbf{x} \\
(-\mathbf{a} + 5\mathbf{x}) - 4\mathbf{x} &= 2\mathbf{a} + (4\mathbf{x} - 4\mathbf{x}) \\
-\mathbf{a} + (5\mathbf{x} - 4\mathbf{x}) &= 2\mathbf{a} + \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
-\mathbf{a} + (5 - 4)\mathbf{x} &= 2\mathbf{a} \\
-\mathbf{a} + (1)\mathbf{x} &= 2\mathbf{a} \\
-\mathbf{a} + \mathbf{x} &= 2\mathbf{a} \\
\mathbf{a} + (-\mathbf{a} + \mathbf{x}) &= \mathbf{a} + 2\mathbf{a} \\
(\mathbf{a} + (-\mathbf{a})) + \mathbf{x} &= (1 + 2)\mathbf{a} \\
\mathbf{0} + \mathbf{x} &= 3\mathbf{a} \\
\mathbf{x} &= 3\mathbf{a}
\end{aligned}
$$

# Linear combinations and coordinates

# Linear combinations and coordinates

**Definition**

A vector $\mathbf{v}$ is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ if there are scalars $c_1, c_2, ..., c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

The scalars $c_1, c_2, ..., c_k$ are called the **coefficients** of the linear combination.

# Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We can use **u** and **v** to locate a new set of axes (in the same way that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ locate the standard coordinate axes). We can use these new axes to determine a **coordinate grid** that will let us easily locate linear combinations of **u** and **v**.

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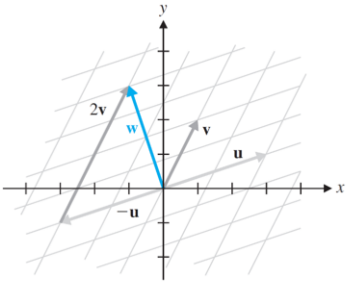

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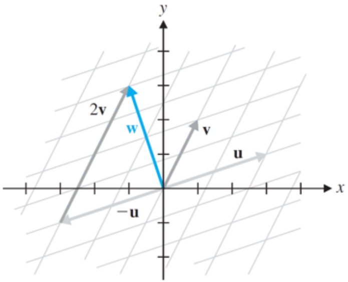


$$\mathbf{w} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

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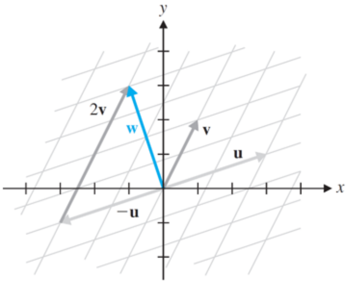


$$\mathbf{w} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(Observe that $-1$ and $3$ are the coordinates of $\mathbf{w}$ with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$.)

# The dot product

# The dot product

The vector versions of length, distance, and angle can all be described using the notion of the dot product of two vectors.

# The dot product

## Definition

If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the **dot product** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$$

# The dot product

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$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$$

Since $\mathbf{u} \cdot \mathbf{v}$ is a number, it is sometimes called the **scalar product** of $\mathbf{u}$ and $\mathbf{v}$.

# The dot product

## Theorem

*Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* $c$ *be a scalar. Then*

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ *(commutativity)*
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w}$ *(distributivity)*
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$
- $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$

# The dot product

Each bullet must be proved. For instance,

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$= v_1u_1 + v_2u_2 + \cdots + v_nu_n = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{v}\cdot\mathbf{u}$$

# Example

Show that $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$

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$$\begin{aligned}
(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}
\end{aligned}$$

# Length

The **length** (or **norm**) of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is the nonnegative scalar defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

# Length

**Theorem**

*Let* $\mathbf{v}$ *be a vector in* $\mathbb{R}^n$ *and let* $c$ *be a scalar. Then*

- $||\mathbf{v}|| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$
- $||c\mathbf{v}|| = |c| \; ||\mathbf{v}||$

# Length

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- $||c\mathbf{v}|| = |c|\ ||\mathbf{v}||$

**Proof.**

(b)

$$\begin{aligned} ||c\mathbf{v}||^2 = c\mathbf{v} \cdot c\mathbf{v} &= c^2v_1^2 + c^2v_2^2 + \cdots + c^2v_n^2 \\ &= c^2(v_1^2 + v_2^2 + \cdots + v_n^2) \\ &= c^2(\mathbf{v} \cdot \mathbf{v}) = c^2||\mathbf{v}||^2 \end{aligned}$$

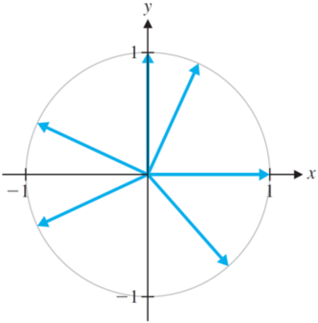
Apply the square root function in both sides

$$||c\mathbf{v}|| = |c|\ ||\mathbf{v}||$$

□

# Length

A vector of length 1 is called a **unit vector**. In $\mathbb{R}^2$, the set of all unit vectors can be identified with the unit circle, the circle of radius 1 centered at the origin.



# Length

Given any nonzero vector $\mathbf{v}$, we can always find a unit vector in the same direction as $\mathbf{v}$ by dividing $\mathbf{v}$ by its own length (or, equivalently, multiplying by $1/||\mathbf{v}||$).

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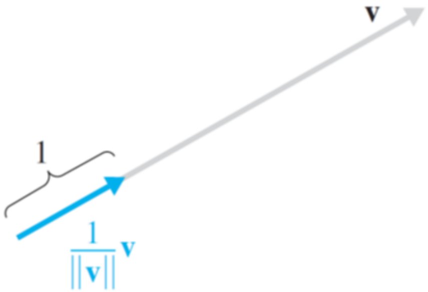
and $\mathbf{u}$ is in the same direction as $\mathbf{v}$, since $1/||\mathbf{v}||$ is a positive scalar.

# Length

Finding a unit vector in the same direction is often referred to as **normalizing** a vector.



# Length

In general, in $\mathbb{R}^n$, we define unit vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$, where $\mathbf{e}_i$ has 1 in its $i$-th component and zeros elsewhere.

These vectors arise repeatedly in linear algebra and are called the **standard unit vectors**.

# Example

Normalize the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

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$$||\mathbf{v}|| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

So, the unit vector in the same direction as $\mathbf{v}$ is given by

$$\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

# Length

**Theorem**

*The Cauchy-Schwarz inequality. For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \ ||\mathbf{v}||$$

**Proof.**

This inequality is equivalent to

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq ||\mathbf{u}||^2 \ ||\mathbf{v}||^2$$

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$$(u_1v_1 + u_2v_2)^2 \leq^? (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$
$$u_1^2v_1^2 + 2u_1v_1u_2v_2 + u_2^2v_2^2 \leq^? u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_2^2v_2^2$$
$$2u_1v_1u_2v_2 \leq^? u_1^2v_2^2 + u_2^2v_1^2$$

# Length

## Proof (cont.)

$$2u_1v_1u_2v_2 \leq^? u_1^2v_2^2 + u_2^2v_1^2$$
$$2(u_1v_2)(u_2v_1) \leq^? (u_1v_2)^2 + (u_2v_1)^2$$

# Length

## Proof (cont.)

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Let $a = u_1v_2$ and $b = u_2v_1$

$$2ab \leq^? a^2 + b^2$$
$$0 \leq^? a^2 + b^2 - 2ab$$

# Length

## Proof (cont.)

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Since

$$a^2 + b^2 - 2ab = (a - b)^2 \geq 0$$

# Length

## Proof (cont.)

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Since

$$a^2 + b^2 - 2ab = (a-b)^2 \geq 0$$

we can remove the “?” sign from all the previous inequalities. (In a conventional style, the proof goes backward). □

# Length

## Theorem

*The triangle inequality. For all vectors* **u** *and* **v** *in* $\mathbb{R}^n$

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

## Proof.

$$\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 &= (u_1+v_1)^2 + \cdots + (u_n+v_n)^2 \\
&= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} \\
&\leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2 \\
&\leq ||\mathbf{u}||^2 + 2||\mathbf{u}||\;||\mathbf{v}|| + ||\mathbf{v}||^2 \\
&= (||\mathbf{u}|| + ||\mathbf{v}||)^2
\end{aligned}$$

# Length

## Theorem

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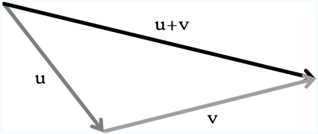
## Proof.

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= (u_1+v_1)^2 + \cdots + (u_n+v_n)^2 \\ &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} \\ &\leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2 \\ &\leq ||\mathbf{u}||^2 + 2||\mathbf{u}|| \; ||\mathbf{v}|| + ||\mathbf{v}||^2 \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$




□

# Distance

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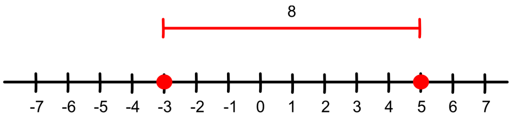

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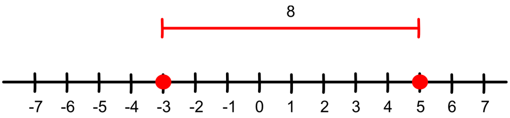


$$d(5, -3) = |5 - (-3)| = | - 3 - 5|$$

# Distance

**Definition**

The **distance** $d(\mathbf{u}, \mathbf{v})$ between vectors **u** and **v** in $\mathbb{R}^n$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

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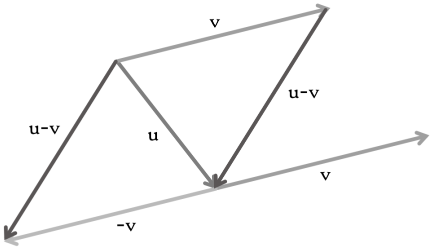

# Example

Find the distance between $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

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Find the distance between $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

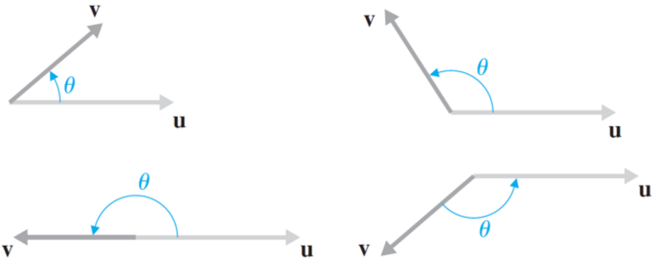
So,

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(\sqrt{2})^2 + (-1)^2 + 1^2} = \sqrt{4} = 2$$

# Angles

# Angles

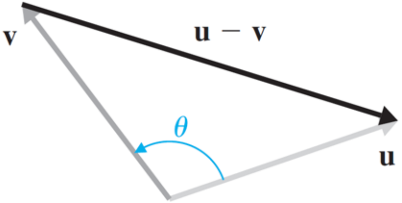
The dot product can also be used to calculate the angle between a pair of vectors. In $\mathbb{R}^2$ or $\mathbb{R}^3$, the angle between the nonzero vector **u** and **v** will refer to the angle $\theta$ determined by these vectors that satisfies $0 \leq \theta \leq 180$.



# Angles

Consider the triangle with sides **u**, **v**, and **u** $-$ **v**, where $\theta$ is the angle between **u** and **v**. Applying the law of cosines to this triangle yields

$$||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$$



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$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||}$$

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For nonzero vectors **u** and **v** in $\mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||}$$

By Cauchy-Schwarz $\left|\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||}\right| \leq 1$. So, $\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||}$ take values between $-1$ and $1$.

# Orthogonal vectors

We now generalize the idea of perpendicularity to vectors in $\mathbb{R}^n$, where it is called **orthogonality**.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, two nonzero vectors **u** and **v** are **perpendicular** if the angle $\theta$ between them is a right angle - that is, if $\theta = \pi/2$ radians, or $90$.

# Orthogonal vectors

Thus,

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||} = \cos 90 = 0$$

and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

# Orthogonal vectors

Thus,

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and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

## Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

# Orthogonal vectors

Thus,

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||} = \cos 90 = 0$$

and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

**Definition**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Since $\mathbf{0} \cdot \mathbf{v}$ for every vector in $\mathbb{R}^n$, the zero vector is orthogonal to every vector.

# Orthogonal vectors

Thus,

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||} = \cos 90 = 0$$

and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

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Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

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Is the zero vector orthogonal to itself?

# Orthogonal vectors

**Theorem**

*Pythagora's theorem. For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$

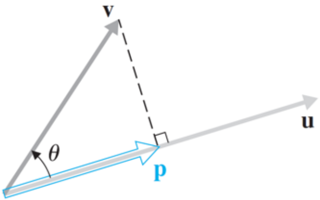
$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

*if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal.*

# Projections

# Projections

Consider two nonzero vectors **u** and **v**. Let **p** be the vector obtained by dropping a perpendicular from the head of **v** onto **u** and let $\theta$ be the angle between **u** and **v**.



# Projections

## Definition

If **u** and **v** are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, the **projection of v onto u** is the vector $proj_{\mathbf{u}(\mathbf{v})}$ defined by

$$proj_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

(You can prove it for $\mathbb{R}^2$)

# Ending

# Homework

- You have three vectors **u**, **v**, and **w** such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Is always $\mathbf{v} = \mathbf{w}$?
- Prove that $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$ (slide 46).
- Prove the Pythagora's theorem for vectors in $\mathbb{R}^n$ (slide 49).
- Prove the definition of projection over $\mathbb{R}^2$ (slide 52).
- $||proj_{\mathbf{u}}(\mathbf{v})|| \leq ||\mathbf{v}||$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Can you **see** why?).
  - Show that this inequality is true in $\mathbb{R}^n$.
  - Show that this inequality is equivalent to the Cauchy-Schwarz inequality.

# Next topics

Matrices

# Thank you