Linear algebra

Vectors

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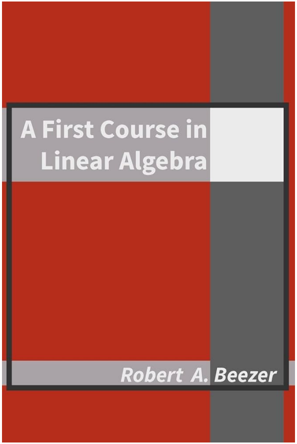

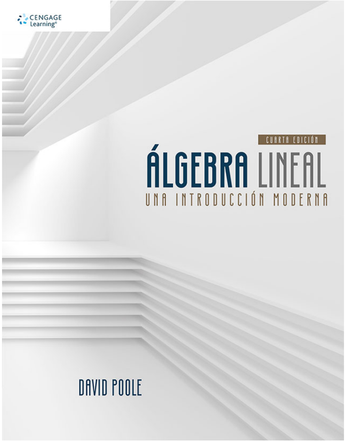

http://linear.ups.edu/

Linear combinations

Linear combinations

Is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$?

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$$\begin{aligned} x-y&=1\\ y&=2\\ 3x-3y&=3 \end{aligned}$$

$$\begin{bmatrix}1&-1&1\\0&1&2\\3&-3&3\end{bmatrix}\xrightarrow{input}\boxed{GJ}\xrightarrow{output}\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}$$

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Is $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

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$$\begin{aligned} x - y &= 2 \\ y &= 3 \\ 3x - 3y &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 3 & -3 & 4 \end{bmatrix} \xrightarrow{input} \boxed{GJ} \xrightarrow{output} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

Theorem

A system of linear equations $\mathcal{LS}(A, \mathbf{b})$ is consistent if and only if **b** *is a linear combination of the columns of A.*

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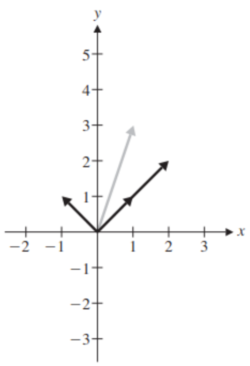

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$$\begin{aligned} x - y &= 2 \\ 2x - 2y &= 4 \end{aligned}$$

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$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

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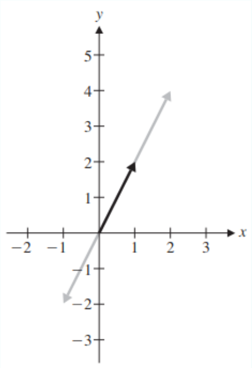

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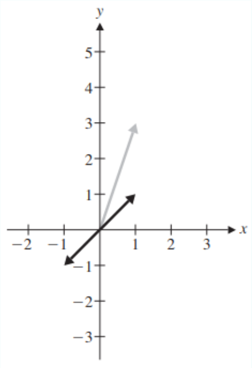

Spanning set

Spanning set

Definition

If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ and is denoted by $\langle\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}\rangle$ or $\langle S\rangle$. If $\langle S\rangle = \mathbb{R}^n$, then S is called a **spanning set** for $\mathbb{R}^n$.

Example

Show that $\mathbb{R}^2 = \langle\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\}\rangle$

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Show that $\mathbb{R}^2 = \langle \{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \} \rangle$

We need to show that an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$; that is, we must show that the equation

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

can always be solved for x and y (in terms of a and b), regardless of the values of a and b.

Example

$$\begin{bmatrix} 2 & 1 & a \\ -1 & 3 & b \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 3 & b \\ 2 & 1 & a \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} -1 & 3 & b \\ 0 & 7 & a+2b \end{bmatrix}$$

Example

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$$\xrightarrow{\frac{1}{7}R_2} \begin{bmatrix} -1 & 3 & b \\ 0 & 1 & (a+2b)/7 \end{bmatrix} \xrightarrow{-3R_2 + R_1} \begin{bmatrix} -1 & 0 & (b-3a)/7 \\ 0 & 1 & (a+2b)/7 \end{bmatrix}$$

Example

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From which we see that $x = (3a - b)/7$ and $y = (a + 2b)/7$. Thus, for any choice of a and b, we have

$$\left(\frac{3a-b}{7}\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \left(\frac{a+2b}{7}\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

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Is $\mathbb{R}^2 = \langle \{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix} \} \rangle$?

Linear dependence

Linear dependence

Definition

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ is **linearly dependent** if there are scalars $c_1, c_2, ..., c_k$, at least one of which is not zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

A set of vectors that is not linearly dependent is called **linearly independent**.

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Are $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ linearly dependent?

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A set of vectors that is not linearly dependent is called **linearly independent**.

Are $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ linearly dependent?

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear dependence

Theorem

Vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *in* $\mathbb{R}^n$ *are linearly dependent if and only at least one of the vectors can be expressed as a linear combination of the others.*

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"Proof"

It is almost obvious by "moving" some linearly dependent vector to the right.

Example 1

Are $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ linearly dependent?

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Is any of them a multiple of the other?

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Are $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ linearly dependent?

Is any of them a multiple of the other?

No. So, they are linearly independent.

Example 2

Are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ linearly dependent?

Example 2

Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ linearly dependent?

Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} + c_3 \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ?$$

Example 2

Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ linearly dependent?

Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} + c_3 \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ?$$

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\1 & 1 & 0 & 0\\0 & 1 & 1 & 0\end{bmatrix} \xrightarrow{input} \boxed{GJ} \xrightarrow{output} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

Example 2

Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ linearly dependent?

Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} + c_3 \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ?$$

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\end{bmatrix} \xrightarrow{input} \boxed{GJ} \xrightarrow{output} \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

Thus, $c_1 = 0, c_2 = 0$, and $c_3 = 0$. The vectors are linearly independent.

Example 3

Are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ linearly dependent?

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Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ?$$

Example 3

Are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ linearly dependent?

Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ?$$

Yes,

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 4

Are $\begin{bmatrix}1\\2\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\4\\2\end{bmatrix}$ linearly dependent?

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Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3 \begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad ?$$

Example 4

Are $\begin{bmatrix}1\\2\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\4\\2\end{bmatrix}$ linearly dependent?

Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\-1\end{bmatrix}+c_3\begin{bmatrix}1\\4\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad ?$$

$$\begin{aligned} c_1+c_2+c_3&=0\\ 2c_1+c_2+4c_3&=0\\ -c_2+2c_3&=0 \end{aligned} \qquad \begin{bmatrix}1&1&1&0\\2&1&4&0\\0&-1&2&0\end{bmatrix} \xrightarrow{input} \boxed{GJ} \xrightarrow{output} \begin{bmatrix}1&0&3&0\\0&1&-2&0\\0&0&0&0\end{bmatrix}$$

Example 4

Are $\begin{bmatrix}1\\2\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\4\\2\end{bmatrix}$ linearly dependent?

Are there scalars c_1, c_2, c_3, different from zero, such that

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\-1\end{bmatrix}+c_3\begin{bmatrix}1\\4\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad ?$$

$$\begin{aligned} c_1+c_2+c_3&=0\\ 2c_1+c_2+4c_3&=0\\ -c_2+2c_3&=0 \end{aligned} \qquad \begin{bmatrix}1&1&1&0\\2&1&4&0\\0&-1&2&0\end{bmatrix}\xrightarrow{input}\boxed{GJ}\xrightarrow{output}\begin{bmatrix}1&0&3&0\\0&1&-2&0\\0&0&0&0\end{bmatrix}$$

Thus, $c_1=-3c_3, c_2=2c_3$, and c_3 is a free variable. Therefore, The vectors are linearly dependent.

Linear dependence

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *be (column) vectors in* $\mathbb{R}^n$ *and let* A *be the* $n \times m$ *matrix* $[\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_m]$ *with these vectors as its columns. Then* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *are linearly dependent if and only if* $\mathcal{LS}(A, \mathbf{0})$ *has a nontrivial solution.*

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Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (column) vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_m]$ with these vectors as its columns. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\mathcal{LS}(A, \mathbf{0})$ has a nontrivial solution.

Proof.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if there are scalars $c_1, c_2, \ldots, c_m$ (not all them zero) such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$.

The previous paragraph is equivalent to saying that the nonzero vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ is a solution of $\mathcal{LS}(A, \mathbf{0})$. □

Example

The standard unit (column) vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent in $\mathbb{R}^3$ because $[\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3 | \mathbf{0}]$ is already in reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It only has the trivial solution. In general, we can see that $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ are linearly independent in $\mathbb{R}^n$

Example (rows)

Row-reduce a matrix with the following **row** vectors

$$[1, 2, 0] \quad [1, 1, -1] \quad [1, 4, 2]$$

Example (rows)

Row-reduce a matrix with the following **row** vectors

$$[1, 2, 0] \quad [1, 1, -1] \quad [1, 4, 2]$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow{R_2' = -R_1 + R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow{R_3' = -R_1 + R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3'' = 2R_2' + R_3'} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example (rows)

Row-reduce a matrix with the following **row** vectors

$$[1, 2, 0] \quad [1, 1, -1] \quad [1, 4, 2]$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow{R_2' = -R_1 + R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow{R_3' = -R_1 + R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3'' = 2R_2' + R_3'} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\mathbf{0} = R_3'' = 2R_2' + R_3' = 2(-R_1 + R_2) + (-R_1 + R_3) = -3R_1 + 2R_2 + R_3$$

Namely,

$$\mathbf{0} = -3[1, 2, 0] + 2[1, 1, -1] + [1, 4, 2]$$

Linear dependence

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *be (row) vectors in* $\mathbb{R}^n$ *and let* A *be the* $m \times n$ *matrix* $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$ *with these vectors as its rows. Then* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ *are linearly dependent if and only if* $r(A) < m$.

Linear dependence

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“Proof”

Just generalize the previous example.

Naming the r number

Definition

The **rank** of a matrix A, denoted by $r(A)$ is the number of nonzero rows in its row-echelon form.

Naming the r number

Theorem

Let A be the coefficient matrix of a system of linear equations with n variables. If the system is consistent, then

$$\textit{number of free variables} = n - r(A)$$

Ending

Summary

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Summary

- Vectors can be the result of **linear combinations** of other vectors.
- A system $\mathcal{LS}(A, \mathbf{b})$ is consistent if and only if **b** is a **linear combination** of the column vectors of A.
- The **span** of a set S of vectors, $\langle\{S\}\rangle$, is the set of all their **linear combinations**.
- A set of vectors is **linearly independent** if none of them is a linear combination of the others. In other words, all vectors are "necessary" or "important" (for the span).

- The column vectors of a matrix A are linearly dependent if and only if $\mathcal{LS}(A, \mathbf{0})$ has a nontrivial solution.

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- The column vectors of a matrix A are linearly dependent if and only if $\mathcal{N}(A)$ has a nonzero vector.
- The row vectors of an $m \times n$ matrix A are linearly dependent if and only if $r(A) < m$.

Homework

Determine if vector **v** is a linear combination of the other vectors.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Homework

Find the span of the following vectors (a) geometrically and (b) algebraically

$$\langle\{\begin{bmatrix}2\\-4\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}\}\rangle \qquad \langle\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}\}\rangle$$

$$\langle\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}3\\2\\-1\end{bmatrix}\}\rangle \qquad \langle\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}\}\rangle$$

Next topics

Matrices

Thank you