Linear algebra

Matrices - 1 of 2

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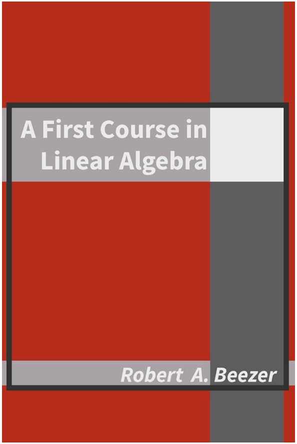
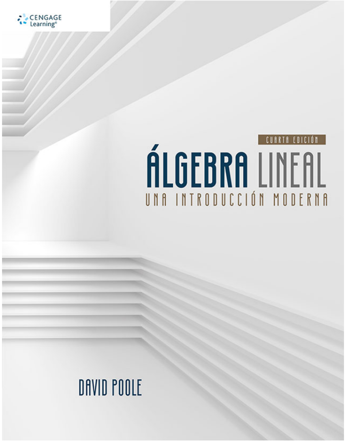
July 9 2024

Contents

1. Matrix operations
2. Matrix addition and scalar multiplication
3. Partitioned matrices
4. Matrix powers
5. The transpose of a matrix
6. Ending

Bibliography





http://linear.ups.edu/

Matrix operations

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A **matrix** is a rectangular array of numbers called the **entries**, or **elements**, of the matrix.

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The **size** of a matrix is a description of the numbers of rows and columns it has. A matrix is called $m \times n$ if it has m rows and n columns.

A $1 \times m$ matrix is called a **row matrix** (or **row vector**), and an $n \times 1$ matrix is called a **column matrix** (or **column vector**).

Matrix operations

We use double-subscript notation to refer to the entries of a matrix A. The entry of A in row i and column j is denoted by a_{ij}.

With this notation, a general $m \times n$ matrix A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix operations

If the columns of A are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$,..., $\mathbf{a}_n$, then we may represent A as

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If the rows of A are $\mathbf{A}_1$, $\mathbf{A}_2$,..., $\mathbf{A}_m$, then we may represent A as

$$A = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}$$

Matrix operations

The **diagonal entries** of A are a_{11}, a_{22}, a_{33},..., and if $m = n$, then A is called a **square matrix**.

A square matrix whose nondiagonal entries are all zero is called a **diagonal matrix**.

A diagonal matrix all of whose diagonal entries are the same is called a **scalar matrix**. If the scalar on the diagonal is 1, the scalar matrix is called an **identity matrix**.

Matrix operations

Two matrices are **equal** if they have the same size and if their corresponding entries are equal.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{r\times s}$, then $A = B$ if and only if $m = r$ and $n = s$ and $a_{ij} = b_{ij}$ for all i and j.

Matrix addition and scalar multiplication

Matrix addition and scalar multiplication

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, their **sum** $A + B$ is the $m \times n$ matrix obtained by adding the corresponding entries. Thus,

$$A + B = [a_{ij} + b_{ij}]$$

If A and B are not the same size, then $A + B$ is not defined.

Matrix addition and scalar multiplication

If A is an $m \times n$ matrix and c is a scalar, the **scalar multiple** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c. More formally, we have

$$cA = c[a_{ij}] = [ca_{ij}]$$

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The matrix $(-1)A$ is written as $-A$ and called the **negative** of A. We can use this fact to define the **difference** of two matrices: If A and B are the same size, then

$$A - B = A + (-B)$$

Matrix addition and scalar multiplication

A matrix all of whose entries are zero is called a **zero matrix** and denoted by O or $O_{m\times n}$.

If A is any matrix and O is the zero matrix of the same size, then

$$A + O = A = O + A$$

and

$$A - A = O = -A + A$$

Matrix multiplication

Definition

If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the **product** $C = AB$ is an $m \times r$ matrix. The (i, j) entry of the product is computed as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{1n}$$

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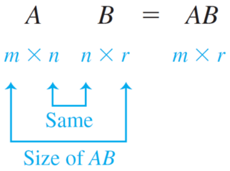
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{1n}$$

Notice that A and B need not be the same size. However, the number of columns of A must be the same as the number of rows of B.



Matrix multiplication

Notice that, in the expression $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{1n}$ the "outer subscripts" on each ab terms are always i and j whereas the "inner subscripts" always agree and increase from 1 to n. We see this pattern clearly if we write c_{ij} using summation notation:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Matrix multiplication

Every linear system can be written in the form $A\mathbf{x} = \mathbf{b}$. In fact, the notation $[A|\mathbf{b}]$ for the augmented matrix of a linear system is just shorthand for the matrix equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 2 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

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Notice that matrix multiplication consists of dot products between rows and columns.

Matrix multiplication

Theorem

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ a $1 \times m$ standard unit vector, and $\mathbf{e}_j$ a $n \times 1$ standard unit vector. Then

(a) $\mathbf{e}_i A$ is the i-th row of A and
(b) $A\mathbf{e}_j$ is the j-th column of A.

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$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Matrix multiplication

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 7 & 8 & 9 & 1 \\ 4 & 5 & 6 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 6 & 9 & 12 & 3 \\ 7 & 8 & 9 & 1 \end{bmatrix}$$

Partitioned matrices

Partitioned matrices

It will often be convenient to regard a matrix as being composed of a number of smaller **submatrices**. By introducing vertical and horizontal lines into a matrix, we can **partition** it into **blocks**.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 7 & 2 \end{bmatrix}$$

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Partitioned matrices

When matrices are being multiplied, there is often an advantage to be gained by viewing them as partitioned matrices.

Suppose A is $m \times n$ and B is $n \times r$, so the product AB exists. If we partition B in terms of its column vectors, as $B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \end{array}\right]$, then

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$$AB = A\left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \end{array}\right] = \left[\begin{array}{c|c|c|c} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_r \end{array}\right]$$

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This result is an immediate consequence of the definition of matrix multiplication. The form on the right is called the **matrix-column representation** of the product.

Partitioned matrices

Suppose A is $m \times n$ and B is $n \times r$, so the product AB exists. If we partition A in terms of its row vectors, as

$$A = \left[\begin{array}{c} \mathbf{A}_1 \\ \hline \mathbf{A}_2 \\ \hline \vdots \\ \hline \mathbf{A}_m \end{array}\right]$$

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Once again, this result is a direct consequence of the definition of matrix multiplication. The form on the right is called the **row-matrix representation** of the product.

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The definition of the matrix product AB uses the natural partition of A into rows and B into columns; this form might be called the **row-column representation** of the product.

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$$AB = \left[\begin{array}{c} \mathbf{A}_1 \\ \hline \mathbf{A}_2 \\ \hline \vdots \\ \hline \mathbf{A}_m \end{array}\right] \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \end{array}\right]$$

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We can also partition A into columns and B into rows; this form is called the **column-row representation** of the product.

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$$AB = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \\ \hline \vdots \\ \hline \mathbf{B}_n \end{array}\right]$$

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Notice that the sum resembles a dot product expansion; the difference is that the individual terms are matrices, not scalars. Each term $\mathbf{a}_i\mathbf{B}i$ is the product of an $m \times 1$ and a $1 \times r$ matrix. Thus, each $\mathbf{a}_i\mathbf{B}_i$ is an $m \times r$ matrix (the same size as AB). The products $\mathbf{a}_i\mathbf{B}_i$ are called outer products, and the sum is called the **outer product expansion** of AB.

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Consider the partitioned matrices A and B as follows.

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$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 & 0 \\ \hline 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 7 & 2 \end{array}\right] \qquad B = \left[\begin{array}{cc|cc|c} 4 & 3 & 1 & 2 & 1 \\ -1 & 2 & 2 & 1 & 1 \\ 1 & -5 & 3 & 3 & 1 \\ \hline 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

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They have the block structures

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

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If we ignore for the moment the fact that their entries are matrices, then A appears to be a 2×2 matrix and B a 2×3 matrix. Their product should thus be a 2×3 matrix given by

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But all of the products in this calculation are actually matrix products, so we need to make sure that they are all defined.

Partitioned matrices

A quick check reveals that this is indeed the case, since the numbers of columns in the blocks of A (3 and 2) match the numbers of rows in the blocks of B. The matrices A and B are said to be **partitioned conformably for block multiplication**.

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Carrying out the calculations indicated gives us the product AB in partitioned form:

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= I_3 B_{11} + A_{12} I_2 \\ &= B_{11} + A_{12} \\ &= \begin{bmatrix} 4 & 3 \\ -1 & 2 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 5 \\ 5 & -5 \end{bmatrix} \end{aligned}$$

Partitioned matrices

The calculations for the other five blocks of AB are similar. Check that the result is

$$\left[\begin{array}{cc|cc|c} 6 & 2 & 1 & 2 & 2 \\ 0 & 5 & 2 & 1 & 12 \\ 5 & -5 & 3 & 3 & 9 \\ \hline 1 & 7 & 0 & 0 & 23 \\ 7 & 2 & 0 & 0 & 20 \end{array}\right]$$

Matrix powers

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When A and B are two $n \times n$ matrices, their product AB will also be an $n \times n$ matrix. A special case occurs when $A = B$. To define $A^2 = AA$ and, in general, to define A^k as

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if k is a positive integer. Thus, $A^1 = A$,
and it is convenient to define $A^0 = I_n$

Matrix powers

The following properties follow immediately from the definitions.

If A is a square matrix and r and s are nonnegative integers, then

- $A^r A^s = A^{r+s}$
- $(A^r)^s = A^{rs}$

Example

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

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and, in general

$$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix} \qquad \text{for all } n \geq 1$$

Example

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then

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The above statement can be proved by mathematical induction, since it is an infinite collection of statements, one for each natural number n.

Example

The basis step is to prove that the formula holds for n $= 1$. In this case,

$$A^1 = \begin{bmatrix} 2^{1-1} & 2^{1-1} \\ 2^{1-1} & 2^{1-1} \end{bmatrix} = \begin{bmatrix} 2^0 & 2^0 \\ 2^0 & 2^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$$

as required.

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as required.

The induction hypothesis is to assume that

$$A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix}$$

for some integer $k \geq 1$. The induction step is to prove that the formula holds for $n = k + 1$

Example

Using the definition of matrix powers and the induction hypothesis, we compute

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$$A^{k+1} = A^k A = \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Example

Using the definition of matrix powers and the induction hypothesis, we compute

$$\begin{aligned} A^{k+1} = A^k A &= \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^{k-1} + 2^{k-1} & 2^{k-1} + 2^{k-1} \\ 2^{k-1} + 2^{k-1} & 2^{k-1} + 2^{k-1} \end{bmatrix} \end{aligned}$$

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Example

Using the definition of matrix powers and the induction hypothesis, we compute

$$
\begin{aligned}
A^{k+1} = A^k A &= \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{k-1} + 2^{k-1} & 2^{k-1} + 2^{k-1} \\ 2^{k-1} + 2^{k-1} & 2^{k-1} + 2^{k-1} \end{bmatrix} \\
&= \begin{bmatrix} 2^{k} & 2^{k} \\ 2^{k} & 2^{k} \end{bmatrix} \\
&= \begin{bmatrix} 2^{(k+1)-1} & 2^{(k+1)-1} \\ 2^{(k+1)-1} & 2^{(k+1)-1} \end{bmatrix}
\end{aligned}
$$

Thus, the formula holds for all $n \geq 1$ by the principle of mathematical induction.

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If $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then

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$$B^4 = B^3 B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$B^4 = B^3 B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^5 = B^4 B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Example

Thus, $B^5 = B$, and the sequence of powers of B repeats in a cycle of four

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ldots$$

The transpose of a matrix

The transpose of a matrix

Definition

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A. that is, the i-th column of A is the i-th row of A^T for all i.

The transpose of a matrix

Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad C = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix}$$

The transpose of a matrix

Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad C = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix}$$

Then their transposes are

$$A^T = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \qquad B^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad C^T = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

The transpose of a matrix

A useful alternative definition of the transpose is given componentwise:

$$(A^T)_{ij} = A_{ji} \qquad \text{for all } i \text{ and } j$$

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$$(A^T)_{ij} = A_{ji} \qquad \text{for all } i \text{ and } j$$

In words, the entry in row i and column j of A^T is the same as the entry in row j and column i of A.

The transpose of a matrix

Definition

A square matrix A is **symmetric** if $A^T = A$ (that is, if A is equal to its own transpose).

A **symmetric** matrix has the property that it is its own "mirror image" across its main diagonal.

A square matrix A is **symmetric** if and only if $A_{ij} = A_{ji}$ for all i and j.

Ending

Summary

- Vocabulary: column matrix (vector), row matrix (vector), square matrix, diagonal matrix, identity matrix, zero matrix, and more.
- Matrix addition, scalar multiplication, matrix multiplication.
- Partitioned matrices.
- Matrix powers.
- The transpose of a matrix.

Homework

- Let
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix}.$$
- Use the matrix-column representation of the product to express each column of AB as a liner combination of the columns of A.
- Use the row-matrix representation of the product to express each row of AB as a linear combination of the rows of B.
- Compute AB using outer products.

Homework

- Use partitioned matrices to find AB, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Find A^0, A^1, A^2,...,A^7. What is A^{2015}?
- Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find a formula for A^n, $n \geq 1$, and verify it using mathematical induction.

Next topics

- **Inverse**
- Transpose
- Determinant

Thank you