Linear algebra

Matrices - 2 of 2

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CONAHCYT
INAOE

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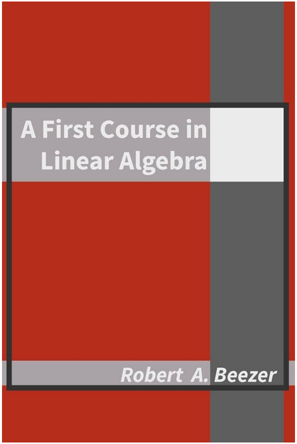

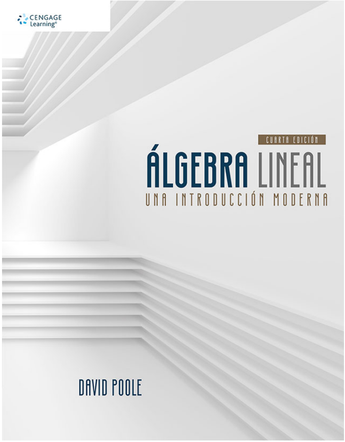

http://linear.ups.edu/

Properties of addition and scalar multiplication

Properties of addition and scalar multiplication

Theorem

Let A, B, and C be matrices of the same size and let c and d be scalar. Then

- $A + B = B + A$ *(commutativity)*
- $(A + B) + C)A + (B + C)$ *(associativity)*
- $A + O = A$
- $A + (-A) = O$
- $c(A + B) = cA + cB$ *(distributivity)*
- $(c + d)A = cA + dA$ *(distributivity)*
- $c(dA) = (cd)A$
- $1A = A$

Properties of addition and scalar multiplication

We can define the **span** of a set of matrices to be the set of all linear combinations of the matrices.

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Describe the span $\langle\{A_1, A_2, A_3\}\rangle$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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One way to do this is simply to write out a general linear combination of A_1, A_2, and A_3. Thus,

$$\begin{aligned} c_1A_1 + c_2A_2 + c_3A_3 &= c_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_2 + c_3 & c_1 + c_3 \\ -c_1 + c_3 & c_2 + c_3 \end{bmatrix} \end{aligned}$$

Properties of addition and scalar multiplication

Notice that a matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is in $\langle\{A_1, A_2, A_3\}\rangle$ when

$$\begin{bmatrix} c_2 + c_3 & c_1 + c_3 \\ -c_1 + c_3 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

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The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & w \\ 1 & 0 & 1 & x \\ -1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right]$$

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The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & w \\ 1 & 0 & 1 & x \\ -1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right]$$

and row reduction produces

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}x - \frac{1}{2}y \\ 0 & 1 & 0 & -\frac{1}{2}x - \frac{1}{2}y + w \\ 0 & 0 & 1 & \frac{1}{2}x + \frac{1}{2}y \\ 0 & 0 & 0 & w - z \end{array}\right]$$

Properties of addition and scalar multiplication

The only restriction comes from the last row, where clearly we must have $w - z = 0$ in order to have a solution. Thus, $\langle\{A_1, A_2, A_3\}\rangle$ consists of all matrices $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ for which $w = z$. That is,

$$\langle\{A_1, A_2, A_3\}\rangle = \left\{ \begin{bmatrix} w & x \\ y & w \end{bmatrix} \;:\; w, x, y \in \mathbb{C} \right\}$$

Properties of addition and scalar multiplication

Note

Linear independence also makes sense for matrices. We say that matrices A_1, A_2,..., A_k of the same size are **linearly independent** if the only solution of the equation

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k = O$$

is the trivial one: $c_1 = c_2 = \cdots = c_k = 0$. If there are nontrivial coefficients that satisfy this equation, then A_1, A_2,..., A_k are called linearly dependent.

Properties of matrix multiplication

Properties of matrix multiplication

Consider the matrices

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

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Thus, $AB \neq BA$. So, in contrast to multiplication of real numbers, matrix multiplication is not commutative (the order of the factors in a product matters!).

Properties of matrix multiplication

Theorem

Let A, B, and C be matrices (whose sizes are such that the indicated operations can be performed) and let k be a scalar. Then

- $A(BC) = (AB)C$ *(associativity)*
- $A(B + C) = AB + AC$ *(left distributivity)*
- $(A + B)C = AC + BC$ *(right distributivity)*
- $k(AB) = (kA)B = A(kB)$
- $I_m A = A = AI_n$ *if* A *is* $m \times n$ *(multiplicative identity)*

Properties of the transpose

Properties of the transpose

Theorem

Let A and B be matrices (whose sizes are such that the indicated operations can be performed) and let k be a scalar. Then

- $(A^T)^T = A$
- $(kA)^T = k(A^T)$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^r)^T = (A^T)^r \quad r \geq 0$

Properties of the transpose

Theorem

$(AB)^T = B^T A^T$

Proof.

Let $row_i(X)$ be the i-th row of matrix X and let $col_j(X)$ be the j-th column of matrix X. Thus,

$$[(AB)^T]_{ij} = [AB]_{ji}$$

□

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$$\begin{aligned}[(AB)^T]_{ij} &= [AB]_{ji} \\ &= row_j(A) \cdot col_i(B)\end{aligned}$$

□

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$(AB)^T = B^T A^T$

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Let $row_i(X)$ be the i-th row of matrix X and let $col_j(X)$ be the j-th column of matrix X. Thus,

$$\begin{aligned}
[(AB)^T]_{ij} &= [AB]_{ji} \\
&= row_j(A) \cdot col_i(B) \\
&= col_j(A^T) \cdot row_i(B^T) \\
&= row_i(B^T) \cdot col_j(A^T) \\
&= [B^T A^T]_{ij}
\end{aligned}$$

□

Some observations

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Then

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Then

$$A + A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$$

Some observations

Let

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Then

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$$BB^T = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 5 \\ 5 & 14 \end{bmatrix}$$

Some observations

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$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Then

$$A + A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$$

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$$B^T B = \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 2 & 2 \\ 2 & 10 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Properties of the transpose

Theorem

(a) If A is a square matrix, then $A + A^T$ is a symmetric matrix.
(b) For any matrix A, AA^T and $A^T A$ are symmetric matrices.

Properties of the transpose

Theorem

(a) If A is a square matrix, then $A + A^T$ is a symmetric matrix.
(b) For any matrix A, AA^T and A^TA are symmetric matrices.

Proof.

(a)

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

Do you remember the definition of a symmetric matrix? □

The inverse of a matrix

The inverse of a matrix

Definition

If A is an $n \times n$ matrix, an inverse of A is an $n \times n$ matrix A' with the property that

$$AA' = I \quad \text{and} \quad A'A = I$$

where $I = I_n$ is the $n \times n$ identity matrix. If such an A' exists, then A is called **invertible**.

The inverse of a matrix

Theorem

If A is an invertible matrix, then its inverse is unique.

The inverse of a matrix

Theorem

If A is an invertible matrix, then its inverse is unique.

Proof.

Temporarily, assume that there are two different inverses

$$A'A = AA' = I$$

$$A''A = AA'' = I$$

Then,

□

The inverse of a matrix

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If A is an invertible matrix, then its inverse is unique.

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$$A''A = AA'' = I$$

Then,

$$A' = IA'$$

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Then,

$$\begin{aligned} A' &= IA' \\ &= A''AA' \end{aligned}$$

□

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Then,

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□

The inverse of a matrix

Theorem

If A is an invertible $n \times n$ matrix, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$ in $\mathbb{R}^n$.

Proof.

$$A\mathbf{x} = \mathbf{b}$$

□

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Proof.

$$A\mathbf{x} = \mathbf{b}$$
$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

□

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Proof.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

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□

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Proof.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

A is invertible and A^{-1} is unique. Therefore

$$\mathbf{x} = A^{-1}\mathbf{b}$$

which is unique. □

"Another" point of view

$$\begin{aligned} 2x + y &= 3 \\ -x + y &= 0 \end{aligned} \equiv \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \end{bmatrix} \equiv \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

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What is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ that, under transformation $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, "lands" on $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$?

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What is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ that, under transformation $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, "lands" on $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$?

where

$$\begin{bmatrix} 2x + y \\ -x + y \end{bmatrix} = \begin{bmatrix} 2x \\ -x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

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What is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ that, under transformation $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, “lands” on $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$?

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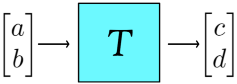
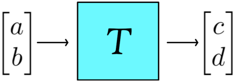
So, “to land” means: to express the vector of constants as a linear combination of the column vectors of the coefficients matrix.

“Another” point of view



“Another” point of view

$$f(a) = b$$

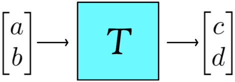


$$T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

"Another" point of view

$$f(a) = b$$

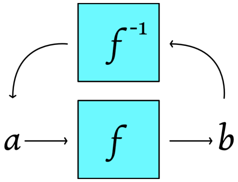
Assume f is bijective.



$$T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

Assume T is invertible.

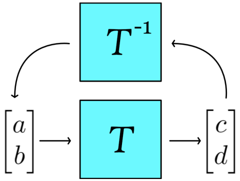
"Another" point of view



$$f(a) = b$$

Assume f is bijective.

$$f^{-1}(f(a)) = f^{-1}(b) = a$$

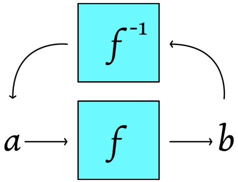


$$T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

Assume T is invertible.

$$T^{-1}T \begin{bmatrix} a \\ b \end{bmatrix} = T^{-1} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

"Another" point of view

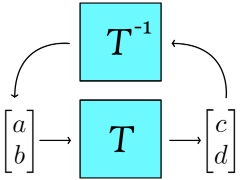


$$f(a) = b$$

Assume f is bijective.

$$f^{-1}(f(a)) = f^{-1}(b) = a$$

$$(f^{-1} \circ f)(a) = a$$



$$T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

Assume T is invertible.

$$T^{-1}T \begin{bmatrix} a \\ b \end{bmatrix} = T^{-1} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

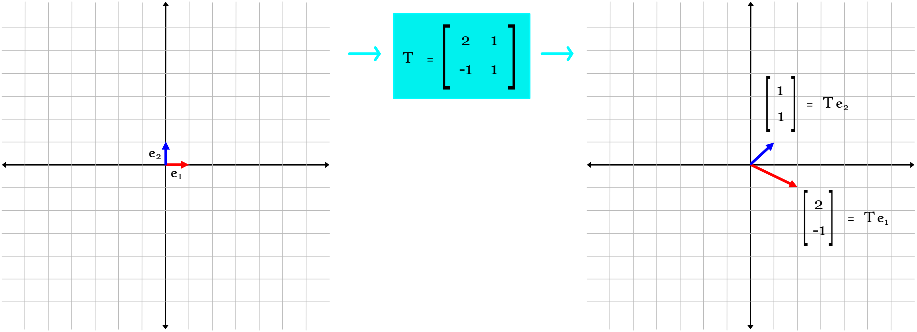
$$T^{-1}T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Example 1

$$\begin{aligned} 2x + y &= 3 \\ -x + y &= 1 \end{aligned}$$

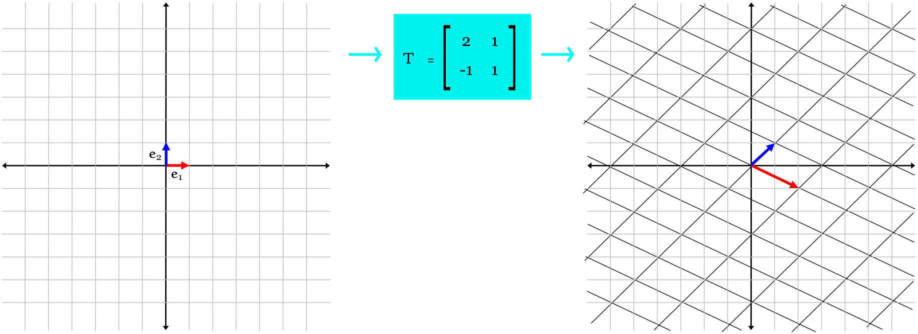
$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example 1



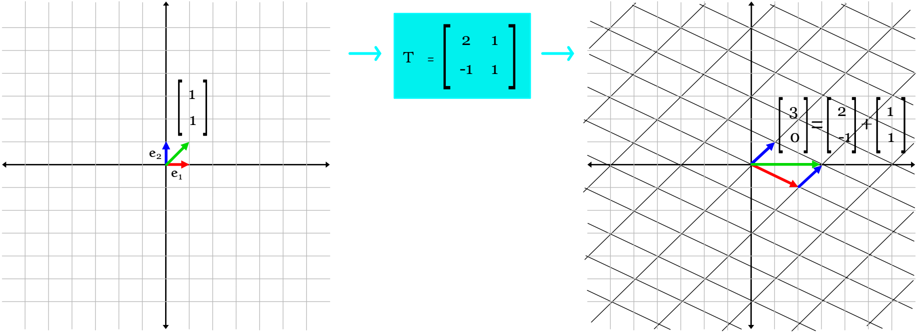
$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad , \quad \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 1



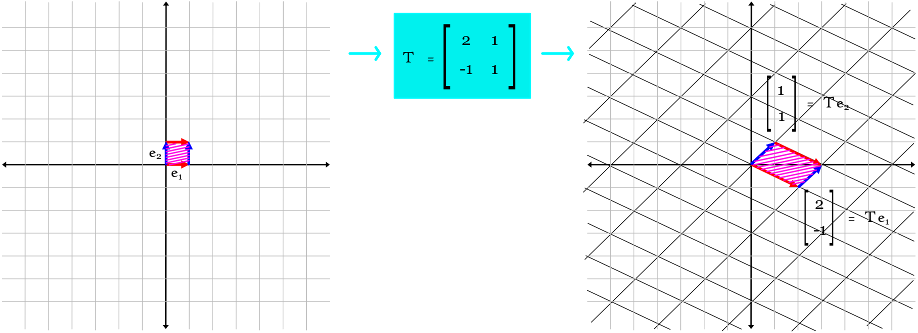
$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ -x + y \end{bmatrix} = \begin{bmatrix} 2x \\ -x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Example 1



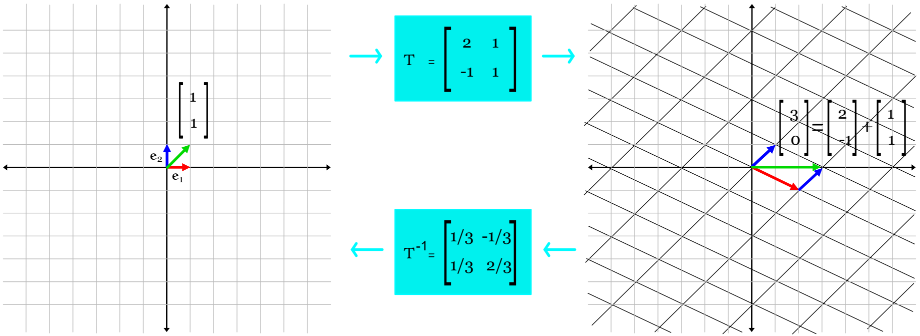
$$\begin{bmatrix}3\\0\end{bmatrix} = 1\begin{bmatrix}2\\-1\end{bmatrix} + 1\begin{bmatrix}1\\1\end{bmatrix}$$

Example 1



$$\det T = \det \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = (2)(1) - (-1)(1) = 2 + 1 = 3$$

Example 1



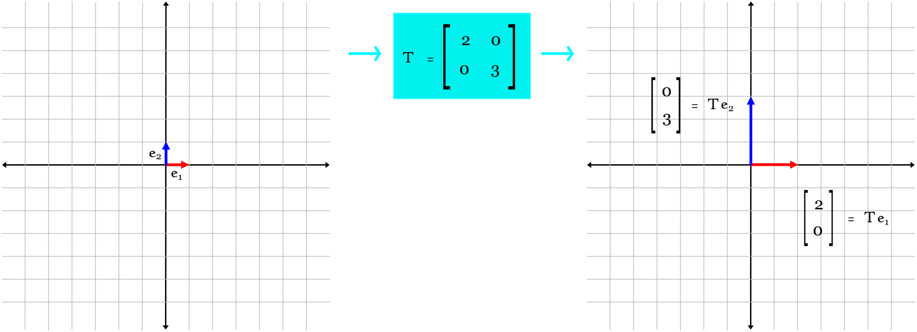
$$\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 2

$$\begin{aligned} 2x &= 3 \\ y &= 0 \end{aligned}$$

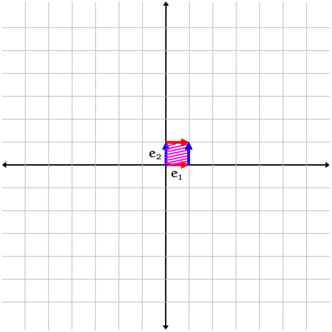
$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

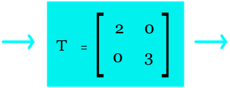
Example 2

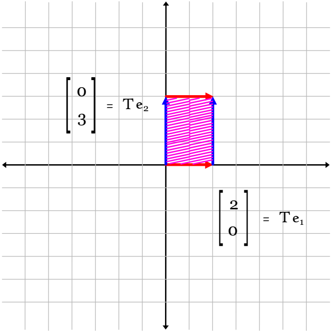


$$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \quad , \quad \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

Example 2

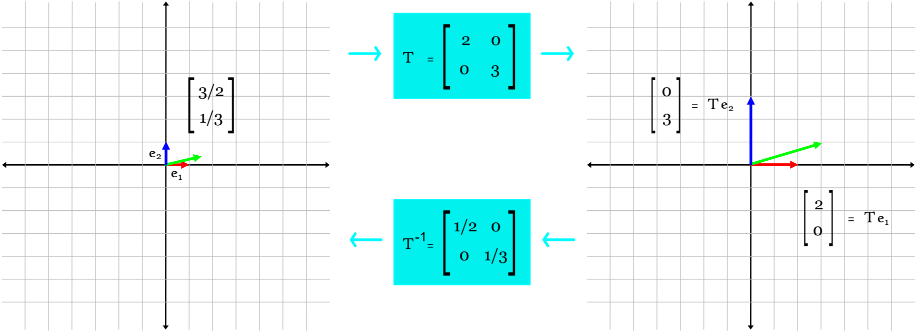






$$\det T = \det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = (2)(3) - (0)(0) = 6 + 0 = 6$$

Example 2



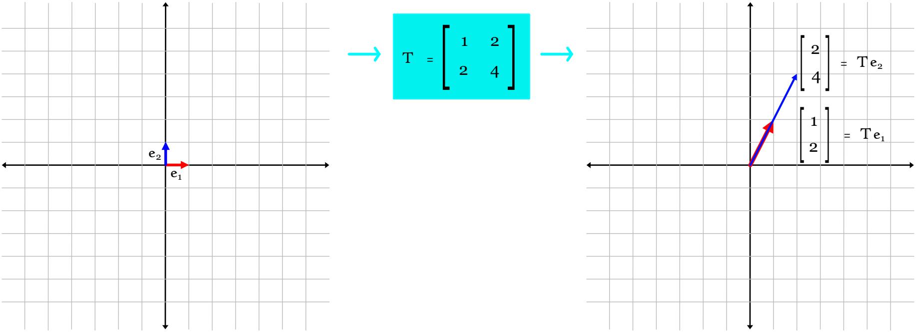
$$\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/3 \end{bmatrix}$$

Example 3

$$\begin{aligned} x + 2y &= -1 \\ 2x + 4y &= -2 \end{aligned}$$

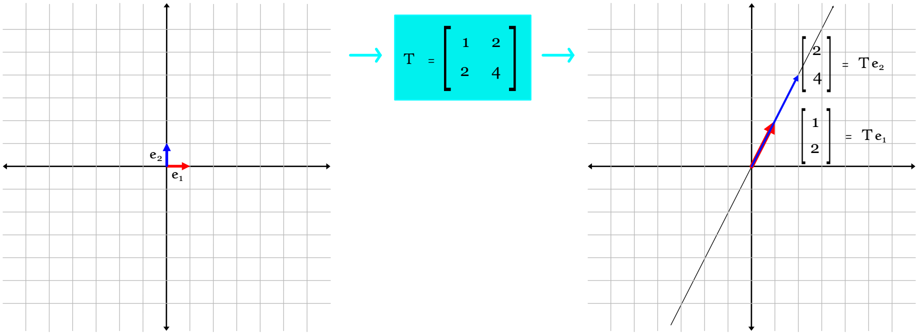
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Example 3



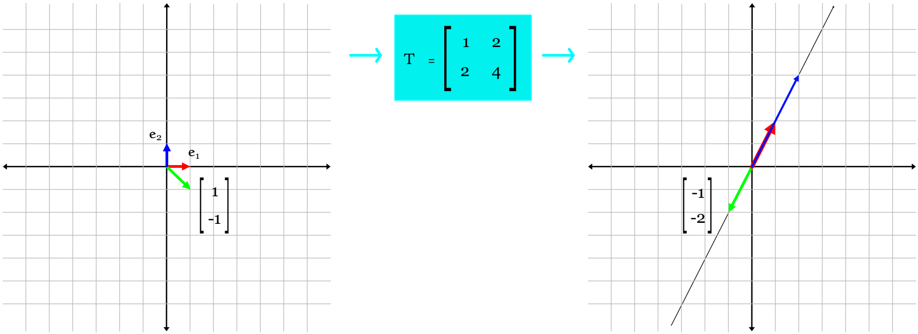
$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix} \quad , \quad \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

Example 3



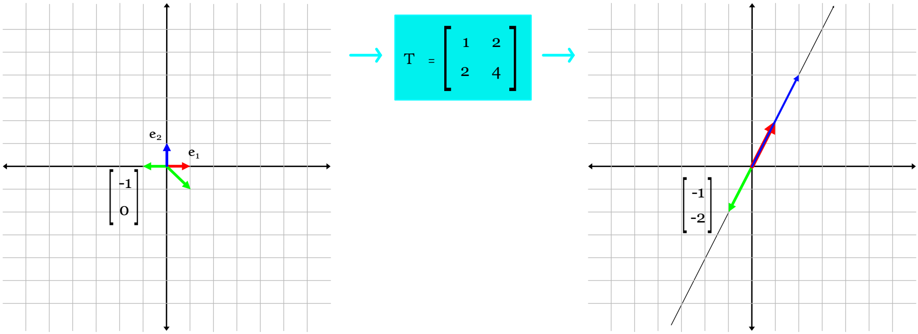
$$\det T = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = (1)(4) - (2)(2) = 4 - 4 = 0$$

Example 3



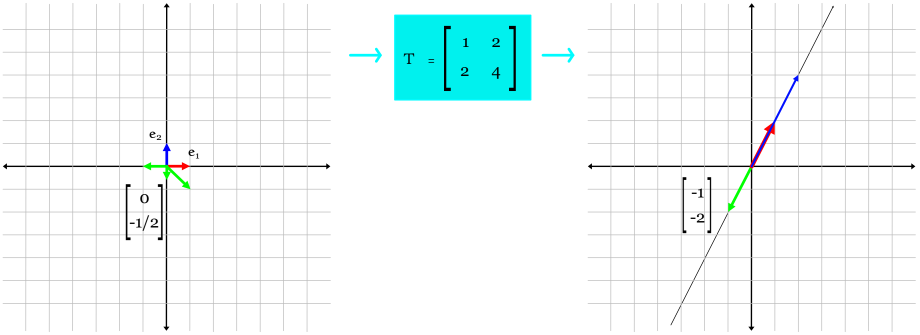
$$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-1\\ -2\end{bmatrix}$$

Example 3



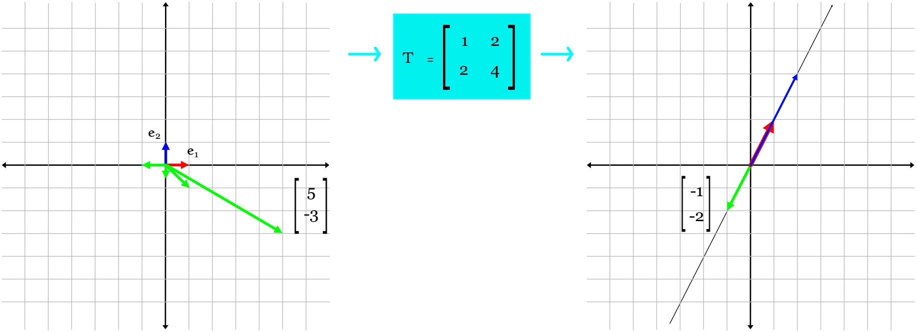
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Example 3



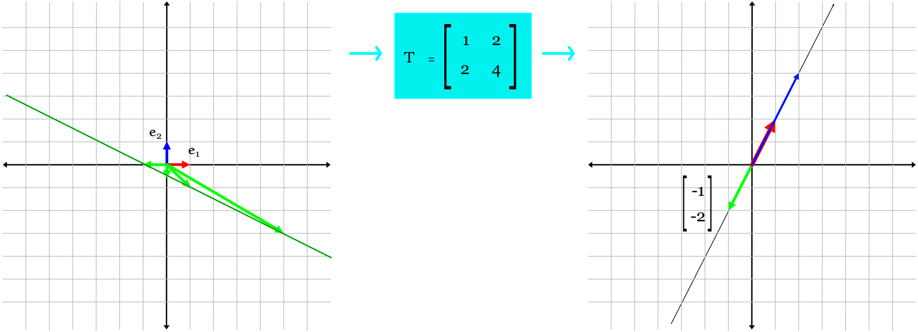
$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}0\\-1/2\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

Example 3



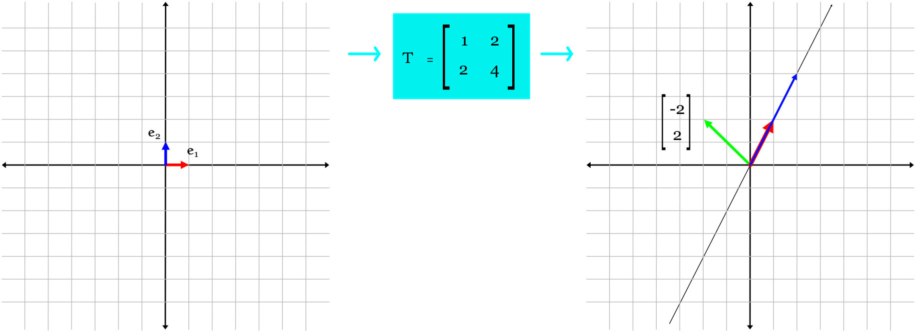
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Example 3



$$S\left(\mathcal{LS}\left(T, \begin{bmatrix} -1 \\ -2 \end{bmatrix}\right)\right) = \left\{ \begin{bmatrix} x \\ (-x/2) - (1/2) \end{bmatrix} \;:\; x \in \mathbb{R} \right\}$$

Example 3



$$S\left(\mathcal{LS}\left(T, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right)\right) = \emptyset$$

The inverse of a matrix

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, then* A *is invertible if* $ad - bc \neq 0$*, in which case*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$*, then* A *is not invertible. The expression* $ad - bc$ *is called the* **determinant** *of* A*, denoted* $\det A$*.*

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Proof.

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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Proof.

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & db - bd \\ -ca + ca & -cb + ad \end{bmatrix}$$

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Proof.

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & db - bd \\ -ca + ca & -cb + ad \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix}$$
$$= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

The inverse of a matrix

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, then* A *is invertible if* $ad - bc \neq 0$*, in which case*

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$$= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Can you see the last step? □

Determinants

Determinants

Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar

$$\det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

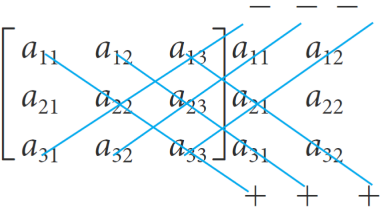
$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

For any square matrix A, $\det A_{ij}$ is called the **(i,j)-minor** of A.

Determinants

Another method for calculating the determinant of a 3×3 matrix is analogous to the method for calculating the determinant of a 2×2 matrix.



Determinants

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the **determinant** of A is the scalar

$$\begin{aligned}\det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

By combining a minor with its plus or minus sign

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

where C_{ij} is called the **(i,j)-cofactor of A**. This way of defining the determinant is often referred to as **cofactor expansion along the first row**.

Properties of invertible matrices

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Properties of invertible matrices

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If A is an invertible matrix, then A^{-1} is invertible and

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Proof.

$$(A^{-1})^{-1}A^{-1} = I$$

□

Properties of invertible matrices

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Proof.

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$$(A^{-1})^{-1} A^{-1} A = IA$$

□

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Proof.

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$$(A^{-1})^{-1}A^{-1}A = IA$$
$$(A^{-1})^{-1}I = A$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

Proof.

$$\begin{aligned}(A^{-1})^{-1}A^{-1} &= I \\ (A^{-1})^{-1}A^{-1}A &= IA \\ (A^{-1})^{-1}I &= A \\ (A^{-1})^{-1} &= A\end{aligned}$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix and c is a nonzero scalar, then cA is invertible matrix and

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

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Proof.

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$$(cA)^{-1}cI = A^{-1}$$
$$cI(cA)^{-1} = A^{-1}$$

Properties of invertible matrices

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Proof.

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(cA)^{-1}cA &= I \\
(cA)^{-1}cAA^{-1} &= IA^{-1} \\
(cA)^{-1}cI &= A^{-1} \\
cI(cA)^{-1} &= A^{-1} \\
c(cA)^{-1} &= A^{-1}
\end{aligned}$$

Properties of invertible matrices

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(cA)^{-1} &= \frac{1}{c}A^{-1}
\end{aligned}$$

□

Properties of invertible matrices

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

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Proof.

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$$(AB)^{-1}ABB^{-1} = IB^{-1}$$

Properties of invertible matrices

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Proof.

$$(AB)^{-1}AB = I$$
$$(AB)^{-1}ABB^{-1} = IB^{-1}$$
$$(AB)^{-1}AI = B^{-1}$$

Properties of invertible matrices

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If A and B are invertible matrices of the same size, then AB is invertible and

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Proof.

$$\begin{aligned}
(AB)^{-1}AB &= I \\
(AB)^{-1}ABB^{-1} &= IB^{-1} \\
(AB)^{-1}AI &= B^{-1} \\
(AB)^{-1}A &= B^{-1}
\end{aligned}$$

Properties of invertible matrices

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If A and B are invertible matrices of the same size, then AB is invertible and

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Proof.

$$\begin{aligned}
(AB)^{-1}AB &= I \\
(AB)^{-1}ABB^{-1} &= IB^{-1} \\
(AB)^{-1}AI &= B^{-1} \\
(AB)^{-1}A &= B^{-1} \\
(AB)^{-1}AA^{-1} &= B^{-1}A^{-1}
\end{aligned}$$

□

Properties of invertible matrices

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If A and B are invertible matrices of the same size, then AB is invertible and

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Proof.

$$\begin{aligned}
(AB)^{-1}AB &= I \\
(AB)^{-1}ABB^{-1} &= IB^{-1} \\
(AB)^{-1}AI &= B^{-1} \\
(AB)^{-1}A &= B^{-1} \\
(AB)^{-1}AA^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1}I &= B^{-1}A^{-1}
\end{aligned}$$

Properties of invertible matrices

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof.

$$\begin{aligned}
(AB)^{-1}AB &= I \\
(AB)^{-1}ABB^{-1} &= IB^{-1} \\
(AB)^{-1}AI &= B^{-1} \\
(AB)^{-1}A &= B^{-1} \\
(AB)^{-1}AA^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1}I &= B^{-1}A^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1}
\end{aligned}$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

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If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

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Proof.

Base case

□

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Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$

□

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If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

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Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$

□

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If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

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Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

$$(A^k)^{-1} = (A^{-1})^k$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

$$(A^k)^{-1} = (A^{-1})^k$$

Induction step

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

$$(A^k)^{-1} = (A^{-1})^k$$

Induction step

$$(A^k)^{-1} = (A^{-1})^k$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

$$(A^k)^{-1} = (A^{-1})^k$$

Induction step

$$(A^k)^{-1} = (A^{-1})^k$$
$$(A^k)^{-1}A^{-1} = (A^{-1})^k A^{-1}$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

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Proof.

Base case

$$(A^1)^{-1} = (A^{-1})^1$$
$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

$$(A^k)^{-1} = (A^{-1})^k$$

Induction step

$$(A^k)^{-1} = (A^{-1})^k$$
$$(A^k)^{-1}A^{-1} = (A^{-1})^k A^{-1}$$
$$(AA^k)^{-1} = (A^{-1})^{k+1}$$

□

Properties of invertible matrices

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If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

Proof.

Base case

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$$(A)^{-1} = (A^{-1})$$
$$A^{-1} = A^{-1}$$

Induction hypothesis

$$(A^k)^{-1} = (A^{-1})^k$$

Induction step

$$(A^k)^{-1} = (A^{-1})^k$$
$$(A^k)^{-1}A^{-1} = (A^{-1})^k A^{-1}$$
$$(AA^k)^{-1} = (A^{-1})^{k+1}$$
$$(A^{k+1})^{-1} = (A^{-1})^{k+1}$$

□

Properties of invertible matrices

Theorem

If A is an invertible matrix, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof.

This is left as homework. □

Properties of invertible matrices

Theorem

If A_1, A_2,..., A_n *are invertible matrices of the same size, then* $A_1 A_2 \cdots A_n$ *is invertible and*

$$(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$$

Proof.

This is left as homework. □

Properties of invertible matrices

Definition

If A is an invertible matrix and n is a positive integer, then A^{-n} is defined by

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}$$

Example

Solve the following matrix equation for X (assuming that the matrices involved are such that all of the indicated operations are defined)

$$A^{-1}(BX)^{-1} = (A^{-1}B^3)^2$$

Example

$$
\begin{aligned}
A^{-1}(BX)^{-1} = (A^{-1}B^3)^2 &\Rightarrow ((BX)A)^{-1} = (A^{-1}B^3)^2 \\
&\Rightarrow [((BX)A)^{-1}]^{-1} = [(A^{-1}B^3)^2]^{-1} \\
&\Rightarrow (BX)A = [(A^{-1}B^3)(A^{-1}B^3)]^{-1} \\
&\Rightarrow (BX)A = B^{-3}(A^{-1})^{-1}B^{-3}(A^{-1})^{-1} \\
&\Rightarrow BXA = B^{-3}AB^{-3}A \\
&\Rightarrow B^{-1}BXAA^{-1} = B^{-1}B^{-3}AB^{-3}AA^{-1} \\
&\Rightarrow IXI = B^{-4}AB^{-3}I \\
&\Rightarrow X = B^{-4}AB^{-3}
\end{aligned}
$$

Elementary matrices

Elementary matrices

Definition

An **elementary matrix** is any matrix that can be obtained by performing an elementary row operation on an identity matrix.

Elementary matrices

Theorem

Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.

Elementary matrices

Theorem

Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.

Proof.

It follows from the definition of matrix multiplication and identity matrix. Since it is kind of intuitive, we omit the proof. □

Elementary matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ i & j & k & l \\ e & f & g & h \end{bmatrix}$$

Example

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then E_1 corresponds to $R_2 \leftrightarrow R_3$, which is undone by doing $R_2 \leftrightarrow R_3$ again.

Thus, $E_1^{-1} = E_1$.

$$E_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then E_2 comes from $4R_2$, which is undone by $\frac{1}{4}R_2$. So,

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Then E_3 comes from $-2R_1 + R_3$, which is undone by $2R_1 + R_3$. So,

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

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Can you see why elementary matrices are invertible? Besides, the inverse of an elementary matrix is invertible too.

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Of course, this comes from how they relate to systems of equations whose augmented matrix has the identity as reduced row-echelon form (so, there are no free variables) and to the notion of nonsingular matrices.

Elementary matrices

Can you see why elementary matrices are invertible? Besides, the inverse of an elementary matrix is invertible too.

Of course, this comes from how they relate to systems of equations whose augmented matrix has the identity as reduced row-echelon form (so, there are no free variables) and to the notion of nonsingular matrices.

Theorem

Each elementary matrix is invertible, and its inverse is an elementary matrix that performs a row-operation of the same type.

The fundamental theorem of invertible matrices (version 1)

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent

- A *is nonsingular.*
- $\mathcal{LS}(A, \mathbf{0})$ *has only the trivial solution.*
- $\mathcal{N}(A)$ *has only the zero vector.*
- $\mathcal{LS}(A, \mathbf{b})$ *has a unique solution for every* $\mathbf{b} \in \mathbb{R}^n$*.*
- A *is invertible.*
- $A\mathbf{x} = \mathbf{b}$ *has a unique solution for every* $\mathbf{b} \in \mathbb{R}^n$*.*
- $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
- *The reduced row-echelon form of* A *is* I_n*.*
- A *is a product of elementary matrices.*

Example

If possible, express $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

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We row reduce A as follows

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} I_2$$

Example

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So, $E_4 E_3 E_2 E_1 A = I$, where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$$

Example

If possible, express $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

We row reduce A as follows

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} I_2$$

So, $E_4 E_3 E_2 E_1 A = I$, where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$$

Therefore,

$$A = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

Invertible matrices

Theorem

Let A be a square matrix. If B is a square matrix such that either $AB = I$ **OR** *$BA = I$, then A is invertible and $B = A^{-1}$.*

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Proof.

Let $BA = I$.

□

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$$A\mathbf{x} = \mathbf{0}$$

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$$BA\mathbf{x} = B\mathbf{0}$$

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$$BA\mathbf{x} = B\mathbf{0}$$
$$I\mathbf{x} = \mathbf{0}$$

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By the fundamental theorem of invertible matrices, A^{-1} exists, so it satisfies $AA^{-1} = I = A^{-1}A$. □

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By the fundamental theorem of invertible matrices, A^{-1} exists, so it satisfies $AA^{-1} = I = A^{-1}A$.

Let us now find A^{-1}.

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By the fundamental theorem of invertible matrices, A^{-1} exists, so it satisfies $AA^{-1} = I = A^{-1}A$.

Let us now find A^{-1}.

$$\begin{aligned} BA &= I \\ BAA^{-1} &= IA^{-1} \end{aligned}$$

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□

Invertible matrices

Cont.

Let $AB = I$.

□

Invertible matrices

Cont.

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$$\mathbf{b} = I\mathbf{b}$$

□

Invertible matrices

Cont.

Let $AB = I$.

$$\begin{aligned} \mathbf{b} &= I\mathbf{b} \\ &= AB\mathbf{b} \end{aligned}$$

□

Invertible matrices

Cont.

Let $AB = I$.

$$\begin{aligned} \mathbf{b} &= I\mathbf{b} \\ &= AB\mathbf{b} \\ &= A(B\mathbf{b}) \end{aligned}$$

□

Invertible matrices

Cont.

Let $AB = I$.

$$\begin{aligned} \mathbf{b} &= I\mathbf{b} \\ &= AB\mathbf{b} \\ &= A(B\mathbf{b}) \end{aligned}$$

So, $\mathbf{x} = B\mathbf{b}$ is the only solution to $A\mathbf{x} = \mathbf{b}$. By the fundamental theorem of invertible matrices, A^{-1} exists, so it satisfies $AA^{-1} = I = A^{-1}A$. □

Invertible matrices

Cont.

Let $AB = I$.

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Let us now find A^{-1}.

$$AB = I$$

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Invertible matrices

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Let us now find A^{-1}.

$$\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1}I \end{aligned}$$

□

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Cont.

Let $AB = I$.

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So, $\mathbf{x} = B\mathbf{b}$ is the only solution to $A\mathbf{x} = \mathbf{b}$. By the fundamental theorem of invertible matrices, A^{-1} exists, so it satisfies $AA^{-1} = I = A^{-1}A$.

Let us now find A^{-1}.

$$\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1}I \\ IB &= A^{-1} \end{aligned}$$

□

Invertible matrices

Cont.

Let $AB = I$.

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So, $\mathbf{x} = B\mathbf{b}$ is the only solution to $A\mathbf{x} = \mathbf{b}$. By the fundamental theorem of invertible matrices, A^{-1} exists, so it satisfies $AA^{-1} = I = A^{-1}A$.

Let us now find A^{-1}.

$$\begin{aligned} AB &= I \\ A^{-1}AB &= A^{-1}I \\ IB &= A^{-1} \\ B &= A^{-1} \end{aligned}$$

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Elementary matrices

Theorem

Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

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Proof.

A is row equivalent to I, then there is a series of k elementary matrices multiplied by the left such that

$$E_k \cdots E_2 E_1 A = I$$

Elementary matrices

Theorem

Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

Proof.

A is row equivalent to I, then there is a series of k elementary matrices multiplied by the left such that

$$E_k \cdots E_2 E_1 A = I$$

By setting $B = E_k \cdots E_2 E_1$, this is $BA = I$. So, by the previous theorem, A is invertible and $A^{-1} = B$. Applying the same sequence of row operations to I is equivalent to multiplying by the left the same sequence of elementary matrices

$$\begin{aligned} E_k \cdots E_2 E_1 I &= BI \\ &= B \\ &= A^{-1} \end{aligned}$$

□

The Gauss-Jordan method for computing the inverse

Gauss-Jordan to find the inverse

Find the inverse of

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

Add the identity to the right and proceed with Gauss-Jordan elimination

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

Gauss-Jordan to find the inverse

$$\xrightarrow{-2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

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$$\xrightarrow{-\frac{1}{2}R_2}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 1 & 1 & -2 & -1 & 0 & 1 \end{array}\right]\xrightarrow{-R_2+R_3}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right]$$

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$$\xrightarrow{R_3+R_1}\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1/2 & 1 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right] \xrightarrow{3R_3+R_2}\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 1/2 & 1 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right]$$

Gauss-Jordan to find the inverse

$$\xrightarrow{-2R_1+R_2}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

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$$\xrightarrow{-2R_2+R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right]$$

Gauss-Jordan to find the inverse

$$\xrightarrow{-2R_1+R_2}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-2&6&-2&1&0\\1&3&-3&0&0&1\end{array}\right]\xrightarrow{-R_1+R_3}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-2&6&-2&1&0\\0&1&-2&-1&0&1\end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&-3&1&-1/2&0\\1&1&-2&-1&0&1\end{array}\right]\xrightarrow{-R_2+R_3}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&-3&1&-1/2&0\\0&0&1&-2&1/2&1\end{array}\right]$$

$$\xrightarrow{R_3+R_1}\left[\begin{array}{ccc|ccc}1&2&0&-1&1/2&1\\0&1&-3&1&-1/2&0\\0&0&1&-2&1/2&1\end{array}\right]\xrightarrow{3R_3+R_2}\left[\begin{array}{ccc|ccc}1&2&0&-1&1/2&1\\0&1&0&-5&1&3\\0&0&1&-2&1/2&1\end{array}\right]$$

$$\xrightarrow{-2R_2+R_1}\left[\begin{array}{ccc|ccc}1&0&0&9&-3/2&-5\\0&1&0&-5&1&3\\0&0&1&-2&1/2&1\end{array}\right]$$

$$\therefore \qquad A^{-1}=\begin{bmatrix}9&-3/2&-5\\-5&1&3\\-2&1/2&1\end{bmatrix}$$

Ending

Summary

- Matrices can be added together (addition) and multiplied by a scalar or between them.
- Matrix multiplication is not commutative.
- If $A^T = A$, the matrix A is symmetric. $A + A^T$, AA^T, and $A^T A$ are symmetric matrices.
- A square matrix A is invertible if $AA' = I = A'A$. A' is its inverse and is unique.
- If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$.

Summary

- Matrices can be seen as transformations (functions) that squishes, expand, rotate, or flip the original space.
- To undo a series of transformations, do them backwards. Namely, $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$.
- If A is a product of elementary matrices, then A is invertible.
- Knowing $AB = I$ **OR** $BA = I$ is enough to guarantee that A is invertible and $B = A^{-1}$.

Homework

- Let
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
Find A^{-1}. Using A^{-1}, solve $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, and $A\mathbf{x} = \mathbf{b}_3$. Finally, use Gauss-Jordan elimination to simultaneously solve the three systems.
- Prove that if A is an invertible matrix, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
- Prove that if a symmetric matrix is invertible, its inverse is symmetric too.
- Prove that $(A^r)^T = (A^T)^r$ for every $r \geq 0$.
- Use mathematical induction to prove that $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$ for every $n \geq 1$.

Next topics

- Subspaces.
- Basis.
- Dimension.
- Rank.

Thank you