Linear algebra
Subspaces, basis, dimension, and rank

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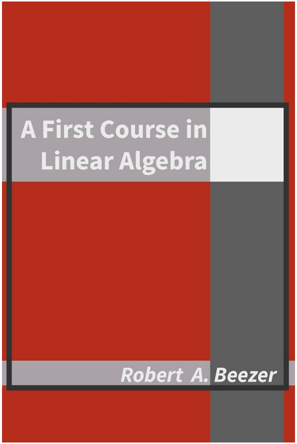
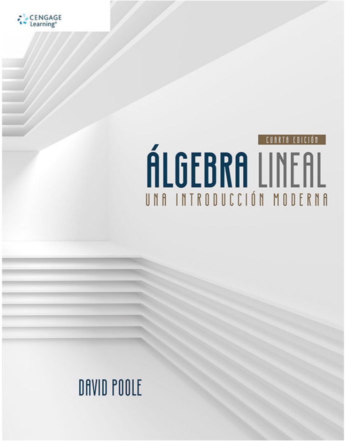
Contents

Bibliography





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Subspaces

Subspaces

Definition

A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:

1. The zero vector $\mathbf{0}$ is in S.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u} + \mathbf{v}$ is in S (the set S is **closed under addition**).
3. If $\mathbf{u}$ is in S and c is an scalar, then $c\mathbf{u}$ is in S (the set S is **closed under scalar multiplication**).

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3. If $\mathbf{u}$ is in S and c is an scalar, then $c\mathbf{u}$ is in S (the set S is **closed under scalar multiplication**).

Notice that properties (2) and (3) can be combined. In this way, S would be **closed under linear combinations**. Namely, If $\mathbf{u}_1$,$\mathbf{u}_2$,...,$\mathbf{u}_k$ are in S and

c_1,c_2,...,c_k are scalars, then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \quad \text{is in} \quad S$$

Example

Every line and plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. It should be clear geometrically that properties (1) through (3) are satisfied. Here is an algebraic proof in the case of a plane through the origin.

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Let $\wp$ be a plane through the origin with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Hence, $\wp = \langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle$. The zero vector $\mathbf{0}$ is in $\wp$, since

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$$

Example

Now, let

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$$

be two vectors in $\wp$. Then,

$$\mathbf{u} + \mathbf{v} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2) = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2$$

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Thus, $\mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so is in $\wp$. Now, let c be a scalar. Then,

$$c\mathbf{u} = c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2$$

which shows that $c\mathbf{u}$ is also a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is therefore in $\wp$. Since $\wp$ satisfies properties (1) through (3), it is a subspace of $\mathbb{R}^3$.

Subspaces

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ *be vectors in* $\mathbb{R}^n$. *Then,* $\langle\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}\rangle$ *is a subspace of* $\mathbb{R}^n$.

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Proof.

Just generalize the previous example. □

Example

Show that the set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfy the condition $x = 3y$ and $z = -2y$ forms a subspace of $\mathbb{R}^3$.

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Substituting the two conditions into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ yields

$$\begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

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Substituting the two conditions into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ yields

$$\begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Since y is arbitrary, the given set of vectors is $\langle\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \}\rangle$ and is thus a subspace of $\mathbb{R}^3$.

Subspaces associated with matrices

Subspaces associated with matrices

Definition

Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace $\mathcal{R}(A)$ of $\mathbb{R}^n$ spanned by the rows of A.
2. The **column space** of A is the subspace $\mathcal{C}(A)$ of $\mathbb{R}^m$ spanned by the columns of A.

Example

Consider the matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$$

Example

Consider the matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$$

1. Determine whether $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the column space of A.
2. Determine whether $\mathbf{w} = \begin{bmatrix} 4 & 5 \end{bmatrix}$ is in the row space of A.
3. Describe $\mathcal{R}(A)$ and $\mathcal{C}(A)$.

Example

(1) We know that **b** is a linear combination of the columns of A if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. We row reduce the augmented matrix as follows:

$$A = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right] \to A = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

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Thus, the system is consistent (and, in fact, has a unique solution). Therefore, **b** is in $\mathcal{C}(A)$.

Example

(2) Elementary row operations simply create linear combinations of the rows of a matrix. That is, they produce vectors only in the row space of the matrix. If the vector **w** is in $\mathcal{R}(A)$, then **w** is a linear combination of the rows of A, so if we augment A by **w** as $\begin{bmatrix} A \\ \hline \mathbf{w} \end{bmatrix}$ it will be possible to apply elementary row operations to this augmented matrix to reduce it to form $\begin{bmatrix} A' \\ \hline \mathbf{0} \end{bmatrix}$ using only elementary row operations of the form $kR_i + R_j$, where $j > i$.

Example

(2) So

$$\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{array}\right] \xrightarrow{-3R_1 + R_3} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 4 & 5 \end{array}\right]$$

$$\xrightarrow{-4R_1 + R_4} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 9 \end{array}\right] \xrightarrow{-9R_2 + R_4} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

Example

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$$\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{array}\right] \xrightarrow{-3R_1 + R_3} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 4 & 5 \end{array}\right]$$

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Therefore, **w** is a linear combination of the rows of A and thus **w** is in $\mathcal{R}(A)$.

Example

(3a) It is easy to check that, for any vector $\mathbf{w} = [x, y]$, the augmented matrix $\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right]$ reduces to

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

Therefore, every vector in $\mathbb{R}^2$ is in $\mathcal{R}(A)$, and so $\mathcal{R}(A) = \mathbb{R}^2$.

Example

(3b) $\mathcal{C}(A) = \langle \{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \} \rangle.$

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So, we are looking for the vectors **x** that satisfy the following equation for any given arbitrary parameters s and t.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

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Namely,

$$\begin{aligned} s - t &= x_1 \\ t &= x_2 \\ 3s - 3t &= x_3 \end{aligned}$$

Example

$$\left[\begin{array}{cc|c} 1 & -1 & x_1 \\ 0 & 1 & x_2 \\ 3 & -3 & x_3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & -3x_1 + x_3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & x_1 + x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & -3x_1 + x_3 \end{array}\right]$$

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So,

$$0 = -3x_1 + x_3$$

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So,

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Thus,

$$\mathcal{C}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 3x_1 \end{bmatrix} \; : \; x_1, x_2 \in \mathbb{R} \right\}$$

Subspaces associated with matrices

Theorem

Let B be any matrix that is row equivalent to a matrix A. Then $\mathcal{R}(B) = \mathcal{R}(A)$.

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Proof.

Since B is row equivalent to A, there is a series of row operations that transforms A into B. Therefore, each row in B is a linear combination of the rows in A. Thus, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

The same applies from B to A. So, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(B)$. □

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First, notice that $A\mathbf{0}_n = \mathbf{0}_m$. So, $\mathbf{0}_n$ is in N.

Now, let $\mathbf{u}$ and $\mathbf{v}$ be in N.

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

which means that $\mathbf{u} + \mathbf{v}$ is in N.

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which means that $\mathbf{u} + \mathbf{v}$ is in N.

Finally, for any scalar c

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}$$

□

Subspaces associated with matrices

Definition

Let A be an $m \times n$ matrix. The **null space** of A is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. It is denoted by $\mathcal{N}(A)$.

Subspaces associated with matrices

Theorem

Let A be a matrix whose entries are real numbers. For any system of linear equations $A\mathbf{x} = \mathbf{b}$, exactly one of the following is true:

- *There is no solution.*
- *There is a unique solution.*
- *There are infinitely many solutions.*

Subspaces associated with matrices

Theorem

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- *There is no solution.*
- *There is a unique solution.*
- *There are infinitely many solutions.*

Proof.

Notice that we want to prove that there cannot be a finite (greater than 1) number of solutions. So, lets see what happens when the system has two different solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. So,

Proof (cont.)

$$A\mathbf{x}_1 = \mathbf{b} \quad \text{and} \quad A\mathbf{x}_2 = \mathbf{b}$$

where $\mathbf{x}_1 \neq \mathbf{x}_2$. Thus,

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Proof (cont.)

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where $\mathbf{x}_1 \neq \mathbf{x}_2$. Thus,

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Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$. So, $\mathbf{x}_0 \neq \mathbf{0}$ and $A\mathbf{x}_0 = \mathbf{0}$. Namely, $\mathcal{N}(A)$ is non trivial. Since $\mathcal{N}(A)$ is closed under scalar multiplication, $c\mathbf{x}_0$ is in $\mathcal{N}(A)$ for any c. Therefore, $\mathcal{N}(A)$ has infinite elements.

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Now, consider all the vectors of the form $\mathbf{x}_1 + c\mathbf{x}_0$, for any c.

$$A(\mathbf{x}_1 + c\mathbf{x}_0) = A\mathbf{x}_1 + cA\mathbf{x}_0 = \mathbf{b} + c\mathbf{0} = \mathbf{b}$$

Thus, there are infinite solutions to $A\mathbf{x} = \mathbf{b}$. □

Basis

Basis

Definition

A **basis** for a subset S of $\mathbb{R}^n$ is a set of vectors in S that

- spans S and
- is linearly independent.

Basis

The standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent and span $\mathbb{R}^n$. Therefore, they form a basis for $\mathbb{R}^n$, called the **standard basis**.

Example

Find a basis for $S = \langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}\rangle$, where

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$$

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The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ already span S, so they will be a basis for S if they are also linearly independent. It is easy to determine that they are not; indeed, $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$.

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The vectors **u**, **v**, and **w** already span S, so they will be a basis for S if they are also linearly independent. It is easy to determine that they are not; indeed, $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$.

Thus, we can ignore **w**, i.e., $\langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}\rangle = \langle\{\mathbf{u}, \mathbf{v}\}\rangle$. Since **u** and **v** are linearly independent, they form a basis for S. (Geometrically, **u**, **v**, and **w** lie within the same plane, and **u** and **u** can serve as direction vectors for this plane.)

Basis

Following is a summary of the most effective procedure to use to find bases for the row space, the column space, and the null space of a matrix A.

- Find the reduced row echelon form R of A.

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Basis

Following is a summary of the most effective procedure to use to find bases for the row space, the column space, and the null space of a matrix A.

- Find the reduced row echelon form R of A.
- Use the nonzero row vectors of R (containing the leading 1s) to form a basis for $\mathcal{R}(A)$.
- Use the column vectors of A that correspond to the columns of R containing the leading 1s (the pivot columns) to form a basis for $\mathcal{C}(A)$.

Basis

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- Find the reduced row echelon form R of A.
- Use the nonzero row vectors of R (containing the leading 1s) to form a basis for $\mathcal{R}(A)$.
- Use the column vectors of A that correspond to the columns of R containing the leading 1s (the pivot columns) to form a basis for $\mathcal{C}(A)$.
- Solve for the leading variables of $R\mathbf{x} = \mathbf{0}$ in terms of the free variables, set the free variables equal to parameters, substitute back into $\mathbf{x}$, and write the result as a linear combination of f vectors (where f is the number of free variables). These f vectors form a basis for $\mathcal{N}(A)$.

Example

Find a basis for $\mathcal{C}(A)$, where

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Let $\mathbf{a}_i$ be a column vector in A and $\mathbf{r}_i$ a column vector in reduced row echelon form.

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Let $\mathbf{a}_i$ be a column vector in A and $\mathbf{r}_i$ a column vector in reduced row echelon form.

Can you see why $\mathbf{r}_3$ is a linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$? Can you see why $\mathbf{r}_4$ is linearly independent from $\mathbf{r}_1$ and $\mathbf{r}_2$?

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Let $\mathbf{a}_i$ be a column vector in A and $\mathbf{r}_i$ a column vector in reduced row echelon form.
Can you see why $\mathbf{r}_3$ is a linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$? Can you see why $\mathbf{r}_4$ is linearly independent from $\mathbf{r}_1$ and $\mathbf{r}_2$?
So, $\mathbf{r}_3$ and $\mathbf{r}_5$ do not contribute to $\mathcal{C}(R)$. Column vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_4$ are linearly independent (they are standard unit vectors). Therefore, a basis for $\mathcal{C}(A)$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Can you see that $\mathcal{C}(A) \neq \mathcal{C}(R)$?

Example

Find a basis for $\mathcal{N}(A)$.

Example

Find a basis for $\mathcal{N}(A)$. Actually, nothing is new here. We are only changing the vocabulary. We must find the solutions to $A\mathbf{x} = \mathbf{0}$. Using the reduced row echelon form

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Find a basis for $\mathcal{N}(A)$. Actually, nothing is new here. We are only changing the vocabulary. We must find the solutions to $A\mathbf{x} = \mathbf{0}$. Using the reduced row echelon form

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+t \\ -2s+3t \\ s \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = s\mathbf{u} + t\mathbf{v}$$

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A basis for $\mathcal{N}(A)$ is $\{\mathbf{u}, \mathbf{v}\}$

Dimension and rank

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Theorem

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Proof.

This is left as homework. □

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Definition

If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted $\dim(S)$.

Dimension and rank

Theorem

The row and column spaces of a matrix A have the same dimension.

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Proof.

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Definition

The rank of a matrix A is the dimension of its row and column spaces and is denoted by $r(A)$.

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Theorem

For any matrix A,

$$r(A^T) = r(A)$$

Dimension and rank

Theorem

For any matrix A,

$$r(A^T) = r(A)$$

Proof.

$$\begin{aligned} r(A^T) &= \dim(\mathcal{C}(A^T)) \\ &= \dim(\mathcal{R}(A)) \\ &= r(A) \end{aligned}$$

□

Dimension and rank

Definition

The nullity of a matrix A is the dimension of its null space and is denoted by $n(A)$.

Dimension and rank

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The nullity of a matrix A is the dimension of its null space and is denoted by $n(A)$.

In other words, $n(A)$ is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which equals the number of free variables in the solution.

Dimension and rank

Theorem (the rank theorem)

If A is an $m \times n$ matrix, then

$$r(A) + n(A) = n$$

Dimension and rank

Theorem (the rank theorem)

If A is an $m \times n$ matrix, then

$$r(A) + n(A) = n$$

Proof.

Let R be the reduced row echelon form of A and let $r(A) = r$. Then, R has r leading 1s. So, there are r dependent variables and $n - r$ free variables in the solution set of $A\mathbf{x} = \mathbf{0}$. Since $\dim(n(A)) = n - r$,

$$\begin{aligned} r(A) + n(A) &= r + (n - r) \\ &= n \end{aligned}$$

□

Example

Find the nullity of :

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{bmatrix}$$

Example

Find the nullity of :

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{bmatrix}$$

Since the two columns of M are clearly linearly independent, $r(M) = 2$. Thus, by the rank theorem,

$$n(M) = 2 - r(M) = 2 - 2 = 0$$

Example

Find the nullity of :

$$N = \begin{bmatrix} 2 & 1 & -2 & -1 \\ 4 & 4 & -3 & 1 \\ 2 & 7 & 1 & 8 \end{bmatrix}$$

Example

Find the nullity of :

$$N = \begin{bmatrix} 2 & 1 & -2 & -1 \\ 4 & 4 & -3 & 1 \\ 2 & 7 & 1 & 8 \end{bmatrix}$$

There is no obvious dependence among the rows or columns of N, so we apply row operations to reduce it to

$$\begin{bmatrix} 2 & 1 & -2 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have reduced the matrix far enough (we do not need reduced row echelon form here, since we are not looking for a basis for the null space). We see that there are only two nonzero rows, so $r(N) = 2$. Hence,

$$n(N) = 4 - r(N) = 4 - 2 = 2$$

The fundamental theorem of invertible matrices (version 2)

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent

- A *is nonsingular.*
- $\mathcal{LS}(A, \mathbf{0})$ *has only the trivial solution.*
- $\mathcal{N}(A)$ *has only the zero vector.*
- $\mathcal{LS}(A, \mathbf{b})$ *has a unique solution for every* $\mathbf{b} \in \mathbb{R}^n$*.*
- A *is invertible.*
- $A\mathbf{x} = \mathbf{b}$ *has a unique solution for every* $\mathbf{b} \in \mathbb{R}^n$*.*
- $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
- *The reduced row-echelon form of* A *is* I_n*.*
- A *is a product of elementary matrices.*

The fundamental theorem of invertible matrices (version 2)

Theorem

More equivalent statements

- $r(A) = n$.
- $n(A) = 0$.
- *The column vectors of A are linearly independent.*
- *The column vectors of A span $\mathbb{R}^n$, i.e., $\mathcal{C}(A) = \mathbb{R}^n$.*
- *The column vectors of A are a basis for $\mathbb{R}^n$.*
- *The row vectors of A are linearly independent.*
- *The row vectors of A span $\mathbb{R}^n$, i.e., $\mathcal{R}(A) = \mathbb{R}^n$.*
- *The row vectors of A are a basis for $\mathbb{R}^n$.*

Ending

Summary

- A **subspace** (of a vector space) contains the zero vector and is closed under linear combinations.
- The **row space** of $A_{m\times n}$, $\mathcal{R}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A.
- The **column space** of $A_{m\times n}$, $\mathcal{C}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
- The **null space** of $A_{m\times n}$, $\mathcal{N}(A)$, is a subspace of $\mathbb{R}^n$.

- A **basis** for a subset S of $\mathbb{R}^n$ is a set of vectors in S that spans S and is linearly independent.
- If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted by $\dim(S)$.
- For any matrix A, $\mathcal{R}(A) = \mathcal{C}(A)$.
- $r(A) + n(A) = n$, where $r(A)$ is the **rank** of the matrix and $n(A)$ is its **nullity**.

Homework

- Let S be the collection of vectors that satisfy the given property. Is S a subspace of $\mathbb{R}^2$?
  1. $x = 0$.
  2. $y = 2x$.
  3. $x \geq 0, \ y \geq 0$.
  4. $xy \geq 0$.
- Let S be the collection of vectors that satisfy the given property. Is S a subspace of $\mathbb{R}^3$?
  1. $x = y = z$.
  2. $z = 2x, \ y = 0$.
  3. $x - y + z = 1$.
  4. $|x - y| = |y - z|$.

Homework

- Prove the theorems from slides 32 and 33.
- Is $\mathbf{b}$ in $\mathcal{C}(A)$? Is $\mathbf{w}$ in $\mathcal{R}(A)$?

1. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$.
2. $A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 & 4 & -5 \end{bmatrix}$.

Homework

- Find a basis for $\mathcal{R}(A)$, $\mathcal{C}(A)$, and $\mathcal{N}(A)$?

1. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$

2. $A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}.$

3. $A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}.$

Homework

- Is $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ a basis for $\mathbb{R}^4$?
- Is $\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$ a basis for $\mathbb{R}^4$?

Next topics

- Eigenvalues and eigenvectors?

Thank you