Exercises - linear algebra - 2024

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1 Trail mix packaging

Exercise 1 Our profit objective function is of the form

$$\text{Profit} = af + b \quad .$$

We know the following:

- $f \in \{825, ..., 960\}$.
- If $a < 0$, the maximum profit is achieved by $f = 825$, which leads to selling 0 kg of the bulk variety.
- If $a > 0$, the maximum profit is achieved by $f = 960$, which leads to selling 0 kg of the standard variety.

Only a coefficient a of 0 allow us to avoid the extreme cases $f = 825$ and $f = 960$. Thus, to sell more than 0 kg of each variety, and still maximize the profit, the coefficient a must be 0. So, we are looking for an x that satisfies the following equation

$$0f + b = (4f - 3300)(4.99 - 3.70) + (-5f + 4800)(x - 3.85) + f(6.50 - 4.45) \quad .$$

By reducing and grouping common terms,

$$\begin{aligned} 0f + b &= 5.16f - 4257 - 5fx + 19.25f - 18480 + 4800x + 2.05f \\ &= 26.46f - 22737 - 5fx + 4800x \\ &= f(26.46 - 5x) - 22737 + 4800x \quad . \end{aligned}$$

Therefore, $26.46 - 5x = 0$ and

$$x = \frac{26.46}{5} = 5.292 \quad .$$

With this value of x, the profit is

$$\text{Profit} = 2664.6 \quad ,$$

which no longer depends on how many kilograms of the fancy variety are sold. Thus, choosing any value in $\{826, ..., 959\}$ implies that more than 0 kilograms of the other varieties will be sold and the profit is maximized. In other words, the objective function is "flat" (has the same value) for any value of f that follows the imposed constraints.

2 Systems of linear equations

Exercise 1 Prove the third equation operation: Multiply each term of one equation by some quantity, and add these terms to a second equation. Leave the first equation the same after this operation, but replace the second equation by the new one.

Let S denote the solution set of the original system, and T denote the solution set of the transformed system. We must prove $S = T$, i.e., (a) $S \subseteq T$ and (b) $T \subseteq S$.

(a) $S \subseteq T$

Suppose α is a number. Let's multiply the terms of equation j by α and add them to equation i. Then, the resulting equation replaces equation i.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
(\alpha a_{j1} + a_{i1})x_1 + (\alpha a_{j2} + a_{i2})x_2 + \cdots + (\alpha a_{jn} + a_{in})x_n &= \alpha b_j + b_i \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Assuming $(x_1, x_2, ..., x_n) = (\beta_1, \beta_2, ..., \beta_n)$ is in S, all equations (different from equation i) in the transformed system are satisfied. So, it is only equation i which requires our attention.

We know that

$$\begin{aligned}
a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{in}\beta_n &= b_i \\
a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n &= b_j \quad .
\end{aligned}$$

So, by distributivity

$$\alpha a_{j1}\beta_1 + \alpha a_{j2}\beta_2 + \cdots + \alpha a_{jn}\beta_n = \alpha b_j$$

By adding equation i and grouping common terms,

$$(\alpha a_{j1} + a_{i1})\beta_1 + (\alpha a_{j2} + a_{i2})\beta_2 + \cdots + (\alpha a_{jn} + \beta_{in})x_n = \alpha b_j + b_i$$

Thus, $S \subseteq T$.

(b) $T \subseteq S$

Assuming $(x_1, x_2, ..., x_n) = (\beta_1, \beta_2, ..., \beta_n)$ is in T, all equations (different from equation i) in the original system are satisfied. So, it is only equation i which requires our attention.

We know that

$$(\alpha a_{j1} + a_{i1})\beta_1 + (\alpha a_{j2} + a_{i2})\beta_2 + \cdots + (\alpha a_{jn} + a_{in})\beta_n = \alpha b_j + b_i \quad .$$

So, by distributivity

$$(\alpha a_{j1}\beta_1 + \alpha a_{j2}\beta_2 + \cdots + \alpha a_{jn}\beta_n) + (a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{i3}\beta_3) = \alpha b_j + b_i$$

By removing, from both sides, $\alpha a_{j1}\beta_1 + \alpha a_{j2}\beta_2 + \cdots + \alpha a_{jn}\beta_n = \alpha b_j$ we get

$$a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{i3}\beta_3 = b_i$$

So, $T \subseteq S$. Since (a) and (b) are true, $S = T$.

Exercise 2 Find all the six-digit numbers in which the first digit is one less than the second, the third digit is half the second, the fourth digit is three times the third, the last two digits sum a number that equals the sum of the fourth and fifth, and the fourth and sixth digits are equal. The sum of all the digits is 24.

First, let us model the problem

$$\begin{aligned}
x_1 &= x_2 - 1 \\
x_3 &= \frac{1}{2}x_2 \\
x_4 &= 3x_3 \\
x_5 + x_6 &= x_4 + x_5 \\
x_4 &= x_6 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 24
\end{aligned}$$

Notice that $x_4 = x_6$. So, we can further reduce the model

$$
\begin{aligned}
x_1 &= x_2 - 1 \\
x_3 &= \frac{1}{2}x_2 \\
x_4 &= 3x_3 \\
\cancel{x_5 + x_4} &= \cancel{x_4 + x_5} \\
\cancel{x_4 = x_4} \\
x_1 + x_2 + x_3 + 2x_4 + x_5 &= 24
\end{aligned}
$$

Thus, this is the resulting augmented matrix of the system of linear equations

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & -1 \\
0 & -1/2 & 1 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 & 24
\end{bmatrix}
$$

Let us apply Gauss-Jordan elimination

$$
\xrightarrow{-R_1 + R_4}
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & -1 \\
0 & -1/2 & 1 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 \\
0 & 2 & 1 & 2 & 1 & 25
\end{bmatrix}
\xrightarrow{-2R_2}
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & -1 \\
0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 \\
0 & 2 & 1 & 2 & 1 & 25
\end{bmatrix}
$$

$$
\xrightarrow{R_2 + R_1}
\begin{bmatrix}
1 & 0 & -2 & 0 & 0 & -1 \\
0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 \\
0 & 2 & 1 & 2 & 1 & 25
\end{bmatrix}
\xrightarrow{-2R_2 + R_4}
\begin{bmatrix}
1 & 0 & -2 & 0 & 0 & -1 \\
0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 \\
0 & 0 & 5 & 2 & 1 & 25
\end{bmatrix}
$$

$$
\xrightarrow{\frac{1}{3}R_3}
\begin{bmatrix}
1 & 0 & -2 & 0 & 0 & -1 \\
0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & -1/3 & 0 & 0 \\
0 & 0 & 5 & 2 & 1 & 25
\end{bmatrix}
\xrightarrow{2R_3 + R_1}
\begin{bmatrix}
1 & 0 & 0 & -2/3 & 0 & -1 \\
0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & -1/3 & 0 & 0 \\
0 & 0 & 5 & 2 & 1 & 25
\end{bmatrix}
$$

$$
\xrightarrow{2R_3 + R_2}
\begin{bmatrix}
1 & 0 & 0 & -2/3 & 0 & -1 \\
0 & 1 & 0 & -2/3 & 0 & 0 \\
0 & 0 & 1 & -1/3 & 0 & 0 \\
0 & 0 & 5 & 2 & 1 & 25
\end{bmatrix}
\xrightarrow{-5R_3 + R_4}
\begin{bmatrix}
1 & 0 & 0 & -2/3 & 0 & -1 \\
0 & 1 & 0 & -2/3 & 0 & 0 \\
0 & 0 & 1 & -1/3 & 0 & 0 \\
0 & 0 & 0 & 11/3 & 1 & 25
\end{bmatrix}
$$

$$
\xrightarrow{\frac{3}{11}R_4}
\begin{bmatrix}
1 & 0 & 0 & -2/3 & 0 & -1 \\
0 & 1 & 0 & -2/3 & 0 & 0 \\
0 & 0 & 1 & -1/3 & 0 & 0 \\
0 & 0 & 0 & 1 & 3/11 & 75/11
\end{bmatrix}
\xrightarrow{\frac{2}{3}R_4 + R_1}
\begin{bmatrix}
1 & 0 & 0 & 0 & 2/11 & 39/11 \\
0 & 1 & 0 & -2/3 & 0 & 0 \\
0 & 0 & 1 & -1/3 & 0 & 0 \\
0 & 0 & 0 & 1 & 3/11 & 75/11
\end{bmatrix}
$$

$$\xrightarrow{\frac{2}{3}R_4+R_2}\begin{bmatrix}1&0&0&0&2/11&39/11\\0&1&0&0&2/11&50/11\\0&0&1&-1/3&0&0\\0&0&0&1&3/11&75/11\end{bmatrix}\xrightarrow{\frac{1}{3}R_4+R_3}\begin{bmatrix}1&0&0&0&2/11&39/11\\0&1&0&0&2/11&50/11\\0&0&1&0&1/11&25/11\\0&0&0&1&3/11&75/11\end{bmatrix}$$

Since the domain of the variables is $\{0, 1, ..., 9\}$, the only value of the free variable x_5 that makes sense is $x_5 = 3$. Therefore, $x_1 = 3$, $x_2 = 4$, $x_3 = 2$, $x_4 = 6$, and $x_6 = 6$. Thus, the six-digit number that satisfies the required conditions is: 342636.

Exercise 3 Suppose that the coefficient matrix of a consistent system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions.

We know that A is consistent and

$$\begin{bmatrix}a_{1i}\\a_{2i}\\\vdots\\a_{mi}\end{bmatrix}=\begin{bmatrix}a_{1j}\\a_{2j}\\\vdots\\a_{mj}\end{bmatrix}$$

where $i < j$. After applying Gauss-Jordan elimination, there are two scenarios: (a) column i is a pivot column or (b) column i is not a pivot column.

Case (a). Since column i is a pivot column, it contains a leading 1 entry, a_{ki}. So, all of its entries are 0 but a_{ki}. Since column i and j are identical, any row operation performed over an entry of column i has the same effect on the corresponding entry of column j. So, by the end of Gauss-Jordan elimination, column j has 0 on all of its entries but in a_{kj}. Since row k has already a leading 1, column j is not a pivot column. Therefore, $j \in F$ and $n - r > 0$. So, there are infinite solutions.

Case (b) is easier. If column i is not a pivot column, then $i \in F$ and $n-r > 0$. So, there are free variables and there are infinite solutions.

Exercise 4 Consider the homogeneous system of linear equations $\mathcal{LS}(A, \mathbf{0})$, and suppose that $\mathbf{u} = [u_1, u_2, ..., u_n]$ is one solution to the system. Prove that $\mathbf{v} = [4u_1, 4u_2, ..., 4u_n]$ is also a solution to $\mathcal{LS}(A, \mathbf{0})$.

Since $\mathbf{u}$ is a solution,

$$
\begin{aligned}
4(a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + \cdots + a_{1n}u_n) &= 4(0)\\
4(a_{21}u_1 + a_{22}u_2 + a_{23}u_3 + \cdots + a_{2n}u_n) &= 4(0)\\
&\vdots\\
4(a_{m1}u_1 + a_{m2}u_2 + a_{m3}u_3 + \cdots + a_{mn}u_n) &= 4(0)
\end{aligned}
$$

By distributivity, commutativity, and associativity

$$
\begin{aligned}
a_{11}4u_1 + a_{12}4u_2 + a_{13}4u_3 + \cdots + a_{1n}4u_n &= 0\\
a_{21}4u_1 + a_{22}4u_2 + a_{23}4u_3 + \cdots + a_{2n}4u_n &= 0\\
&\vdots\\
a_{m1}4u_1 + a_{m2}4u_2 + a_{m3}4u_3 + \cdots + a_{mn}4u_n &= 0
\end{aligned}
$$

Therefore, $\mathbf{v} = [4u_1, 4u_2, ..., 4u_n]$ is also a solution to $\mathcal{LS}(A, \mathbf{0})$.

Exercise 5 If any, find the values of k for which each system has (a) no solution, (b) one solution, (c) infinite solutions. (Avoid using the determinant concept.)

(1)

$$
\begin{aligned}
kx + 2y &= 3\\
2x - 4y &= -6
\end{aligned}
$$

$$
\begin{bmatrix} k & 2 & 3\\ 2 & -4 & -6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -4 & -6\\ k & 2 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & -3\\ k & 2 & 3 \end{bmatrix} \xrightarrow{-kR_1 + R_2} \begin{bmatrix} 1 & -2 & -3\\ 0 & 2k+2 & 3k+3 \end{bmatrix}
$$

$$
\xrightarrow{\frac{1}{2k+2}R_2} \begin{bmatrix} 1 & -2 & -3\\ 0 & 1 & \frac{3k+3}{2k+2} \end{bmatrix} \xrightarrow{2R_2 + R_1} \begin{bmatrix} 1 & 0 & \frac{6k+6}{2k+2} - 3\\ 0 & 1 & \frac{3k+3}{2k+2} \end{bmatrix}
$$

The coefficient matrix is reduced to the identity matrix if and only if $2k+2 \neq 0$. So, there is one solution to $\mathcal{LS}(A, \mathbf{b})$ when $k \neq -1$. In case $k = -1$,

$$
\begin{bmatrix} -1 & 2 & 3\\ 2 & -4 & -6 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & -2 & -3\\ 2 & -4 & -6 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & -2 & -3\\ 0 & 0 & 0 \end{bmatrix}
$$

The last row is all zeros. Therefore, $r = 1$ and there is $n - r = 2 - 1 = 1$ free variable. Thus, there are infinite solutions when $k = -1$. Since all values of k have been explored, there are no values of k that makes the system inconsistent.

(2)

$$
\begin{aligned}
x + ky &= 1 \\
kx + y &= 1
\end{aligned}
$$

$$
\begin{bmatrix} 1 & k & 1 \\ k & 1 & 1 \end{bmatrix} \xrightarrow{-kR_1 + R_2} \begin{bmatrix} 1 & k & 1 \\ 0 & -k^2+1 & -k+1 \end{bmatrix} \xrightarrow{\frac{1}{-k^2+1}R_2} \begin{bmatrix} 1 & k & 1 \\ 0 & 1 & \frac{-k+1}{-k^2+1} \end{bmatrix} \xrightarrow{-kR_2 + R_1} \begin{bmatrix} 1 & 0 & \frac{k^2-k}{1-k^2}+1 \\ 0 & 1 & \frac{1-k}{1-k^2} \end{bmatrix}
$$

So, the coefficient matrix reduces to the identity matrix only if $1 - k^2 \neq 0$, i.e., if $k \neq 1$ and $k \neq -1$. In that case, there is one solution to the system. In case $k = 1$

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

Since the last row is all zeros, there are free variables and the system has infinite solutions.

In case $k = -1$,

$$
\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}
$$

Since the last row corresponds to an unfeasible equation, there is no solution to the system.

3 Vectors

Exercise 1 You have three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Is always $\mathbf{v} = \mathbf{w}$?

Answer

There are vectors $\mathbf{u} = [1, 1]$, $\mathbf{v} = [1, 2]$, and $\mathbf{w} = [2, 1]$ such that

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (1)(1) + (1)(2) = 3 \quad , \\
\mathbf{u} \cdot \mathbf{w} &= (1)(2) + (1)(1) = 3 \quad ,
\end{aligned}
$$

and

$$
\mathbf{v} \neq \mathbf{w} \quad .
$$

Thus, the answer is no.

Exercise 2 Prove that $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$ (slide 46).

Answer

The law of cosines,

$$||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \quad ,$$

is reduced as follows:

$$\begin{aligned} (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) - 2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \\ (\mathbf{u} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) - 2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \\ -(\mathbf{v} \cdot \mathbf{u}) - (\mathbf{u} \cdot \mathbf{v}) &= -2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \\ -2(\mathbf{v} \cdot \mathbf{u}) &= -2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \\ \mathbf{u} \cdot \mathbf{v} &= ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta \quad . \end{aligned}$$

Exercise 3 Prove the Pythagora's theorem for vectors in $\mathbb{R}^n$ (slide 49).

Answer

We must prove the following biconditional

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \quad \leftrightarrow \quad \mathbf{u} \cdot \mathbf{v} = 0 \quad .$$

($\Rightarrow$)

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \\ (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) \\ (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{u}) + (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) \\ (\mathbf{v} \cdot \mathbf{u}) + (\mathbf{u} \cdot \mathbf{v}) &= 0 \\ 2(\mathbf{u} \cdot \mathbf{v}) &= 0 \\ \mathbf{u} \cdot \mathbf{v} &= 0 \end{aligned}$$

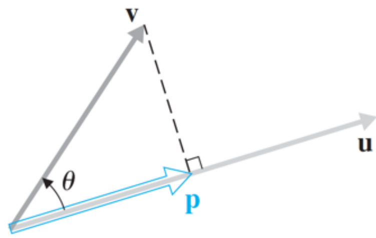
($\Leftarrow$) Do the previous proof backwards.

Exercise 4 Prove the definition of projection over $\mathbb{R}^2$.

Answer

The vector $\mathbf{p}$ we are looking for has length $||\mathbf{p}||$ and the same direction as $\mathbf{u}$. So,

$$\mathbf{p} = ||\mathbf{p}|| \left(\frac{1}{||\mathbf{u}||} \right) \mathbf{u}$$



By trigonometry,

$$\cos\theta = \frac{||\mathbf{p}||}{||\mathbf{v}||}$$

and we know that

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

Thus,

$$\begin{aligned}
\mathbf{p} &= ||\mathbf{v}||\cos\theta\left(\frac{1}{||\mathbf{u}||}\right)\mathbf{u} \\
&= ||\mathbf{v}||\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\right)\left(\frac{1}{||\mathbf{u}||}\right)\mathbf{u} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}\right)\mathbf{u} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}
\end{aligned}$$

So,

$$proj_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

□

Exercise 5 $||proj_{\mathbf{u}}(\mathbf{v})|| \leq ||\mathbf{v}||$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Can you **see** why?). (a) Show that this inequality is true in $\mathbb{R}^n$. (b) Show that this inequality is equivalent to the Cauchy-Schwarz inequality.

Answer

It is evident that, on $\mathbb{R}^2$ and $\mathbb{R}^3$, a projected vector cannot be larger than itself.

(a) By definition,

$$||proj_{\mathbf{u}}(\mathbf{v})|| = ||\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}|| \quad .$$

We know that

$$|\mathbf{u}\cdot\mathbf{v}| \leq ||\mathbf{u}||\ ||\mathbf{v}|| \quad .$$

So,

$$\begin{aligned}
||\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}|| = |\mathbf{u}\cdot\mathbf{v}|\left(\frac{1}{\mathbf{u}\cdot\mathbf{u}}\right)||\mathbf{u}|| &\leq ||\mathbf{u}||\ ||\mathbf{v}||\left(\frac{1}{\mathbf{u}\cdot\mathbf{u}}\right)||\mathbf{u}|| \\
&= ||\mathbf{u}||^2\ ||\mathbf{v}||\left(\frac{1}{\mathbf{u}\cdot\mathbf{u}}\right) \\
&= ||\mathbf{v}||\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right) \\
&= ||\mathbf{v}||
\end{aligned}$$

Therefore, $||proj_{\mathbf{u}}(\mathbf{v})|| \leq ||\mathbf{v}||$.
□

(b) We want to prove that $||proj_{\mathbf{u}}(\mathbf{v})|| \leq ||\mathbf{v}||$ if and only if $|\mathbf{u}\cdot\mathbf{u}| \leq ||\mathbf{u}||\ ||\mathbf{v}||$.

($\Rightarrow$) Our premise is that $||proj_{\mathbf{u}}(\mathbf{v})|| \leq ||\mathbf{v}||$. Since $||proj_{\mathbf{u}}(\mathbf{v})|| = ||\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}||$,

$$\begin{aligned}
||\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}|| &\leq ||\mathbf{v}|| \\
|\mathbf{u}\cdot\mathbf{v}|\left(\frac{1}{\mathbf{u}\cdot\mathbf{u}}\right)||\mathbf{u}|| &\leq ||\mathbf{v}|| \\
|\mathbf{u}\cdot\mathbf{v}|\left(\frac{1}{\mathbf{u}\cdot\mathbf{u}}\right)||\mathbf{u}||^2 &\leq ||\mathbf{v}||\ ||\mathbf{u}|| \\
|\mathbf{u}\cdot\mathbf{v}|\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right) &\leq ||\mathbf{v}||\ ||\mathbf{u}|| \\
|\mathbf{u}\cdot\mathbf{v}| &\leq ||\mathbf{u}||\ ||\mathbf{v}||
\end{aligned}$$

□

($\Leftarrow$) The second part of the proof is like the previous one but backward.

Exercise 6 Determine if vector $\mathbf{v}$ is a linear combination of the other vectors.

(a)

$$\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix} \quad , \quad \mathbf{u}_1 = \begin{bmatrix}1\\-1\end{bmatrix} \quad , \quad \mathbf{u}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Yes, $\mathbf{v}$ is linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1 - 2(3)) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b)

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \quad , \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes, $\mathbf{v}$ is linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(c)

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad , \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

Since the system has no solution, vector $\mathbf{v}$ is not a linear combination of vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

Exercise 7 Find the span of the following vectors.

(a)

$$\langle \{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} , \begin{bmatrix} -1 \\ 2 \end{bmatrix} \} \rangle$$

$$\begin{bmatrix} 2 & -1 & b_1 \\ -4 & 2 & b_2 \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} 2 & -1 & b_1 \\ 0 & 0 & 2b_1 + b_2 \end{bmatrix}$$

The system is consistent only when $2b_1 + b_2 = 0$. Namely, is only for those values that the vector of constants is linear combination of the column vectors. Thus,

$$\langle\{\begin{bmatrix}2\\-4\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}\}\rangle=\{\begin{bmatrix}b_1\\b_2\end{bmatrix}\ :\ 2b_1+b_2=0\}$$

(b)

$$\langle\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}\}\rangle$$

$$\begin{bmatrix}0&3&b_1\\0&4&b_2\end{bmatrix}\xrightarrow{\frac{1}{3}R_1}\begin{bmatrix}0&1&\frac{1}{3}b_1\\0&4&b_2\end{bmatrix}\xrightarrow{-4R_1+R_2}\begin{bmatrix}0&1&\frac{1}{3}b_1\\0&0&-\frac{4}{3}b_1+b_2\end{bmatrix}$$

The system is consistent only when $-\frac{4}{3}b_1+b_2=0$. Namely, is only for those values that the vector of constants is linear combination of the column vectors. Thus,

$$\langle\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}\}\rangle=\{\begin{bmatrix}b_1\\b_2\end{bmatrix}\ :\ -\frac{4}{3}b_1+b_2=0\}$$

(c)

$$\langle\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}3\\2\\-1\end{bmatrix}\}\rangle$$

$$\begin{bmatrix}1&3&b_1\\2&2&b_2\\0&-1&b_3\end{bmatrix}\xrightarrow{-2R_1+R_2}\begin{bmatrix}1&3&b_1\\0&-4&-2b_1+b_2\\0&-1&b_3\end{bmatrix}\xrightarrow{-\frac{1}{4}R_2+R_3}\begin{bmatrix}1&3&b_1\\0&-4&-2b_1+b_2\\0&0&\frac{1}{2}b_1-\frac{1}{4}b_2+b_3\end{bmatrix}$$

The system is consistent only when $\frac{1}{2}b_1-\frac{1}{4}b_2+b_3=0$. Namely, is only for those values that the vector of constants is linear combination of the column vectors. Thus,

$$\langle\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}3\\2\\-1\end{bmatrix}\}\rangle=\{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}\ :\ \frac{1}{2}b_1-\frac{1}{4}b_2+b_3=0\}$$

(d)

$$\langle\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}\}\rangle$$

$$\begin{bmatrix}1&-1&0&b_1\\0&1&-1&b_2\\-1&0&1&b_3\end{bmatrix}\xrightarrow{R_1+R_3}\begin{bmatrix}1&-1&0&b_1\\0&1&-1&b_2\\0&-1&1&b_1+b_3\end{bmatrix}\xrightarrow{R_2+R_3}\begin{bmatrix}1&-1&0&b_1\\0&1&-1&b_2\\0&0&0&b_1+b_2+b_3\end{bmatrix}$$

The system is consistent only when $b_1 + b_2 + b_3 = 0$. Namely, is only for those values that the vector of constants is linear combination of the column vectors. Thus,

$$\langle\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\}\rangle = \{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_1 + b_2 + b_3 = 0\}$$

4 Matrices

Exercise 1 Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix}.$$

(a) Use the matrix-column representation of the product to express each column of AB as a liner combination of the columns of A.

(b) Use the row-matrix representation of the product to express each row of AB as a linear combination of the rows of B.

(c) Compute AB using outer products.

(a)

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$$

where,

$$A\mathbf{b}_1 = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix}$$

$$A\mathbf{b}_1 = 3\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ -4 \\ 0 \end{bmatrix}$$

$$A\mathbf{b}_1 = 0\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ -4 \end{bmatrix}$$

(b)

$$AB = \begin{bmatrix} \mathbf{A}_1 B \\ \mathbf{A}_2 B \\ \mathbf{A}_3 B \end{bmatrix}$$

where,

$$\mathbf{A}_1 B = 1\begin{bmatrix}2 & 3 & 0\end{bmatrix} + 0\begin{bmatrix}1 & -1 & 1\end{bmatrix} + (-2)\begin{bmatrix}-1 & 6 & 4\end{bmatrix} = \begin{bmatrix}4 & -9 & -8\end{bmatrix}$$

$$\mathbf{A}_2 B = (-3)\begin{bmatrix}2 & 3 & 0\end{bmatrix} + 1\begin{bmatrix}1 & -1 & 1\end{bmatrix} + 1\begin{bmatrix}-1 & 6 & 4\end{bmatrix} = \begin{bmatrix}-6 & -4 & 5\end{bmatrix}$$

$$\mathbf{A}_3 B = 2\begin{bmatrix}2 & 3 & 0\end{bmatrix} + 0\begin{bmatrix}1 & -1 & 1\end{bmatrix} + (-1)\begin{bmatrix}-1 & 6 & 4\end{bmatrix} = \begin{bmatrix}5 & 0 & -4\end{bmatrix}$$

(c)

$$AB = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\\ \mathbf{B}_3\end{bmatrix} = \mathbf{a}_1\mathbf{B}_1 + \mathbf{a}_2\mathbf{B}_2 + \mathbf{a}_3\mathbf{B}_3$$

where,

$$\mathbf{a}_1\mathbf{B}_1 = \begin{bmatrix}2 & 3 & 0\\ -6 & -9 & 0\\ 4 & 6 & 0\end{bmatrix}$$

$$\mathbf{a}_2\mathbf{B}_2 = \begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{a}_3\mathbf{B}_3 = \begin{bmatrix}2 & -12 & -8\\ -1 & 6 & 4\\ 1 & -6 & -4\end{bmatrix}$$

So,

$$AB = \begin{bmatrix}2 & 3 & 0\\ -6 & -9 & 0\\ 4 & 6 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 1\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}2 & -12 & -8\\ -1 & 6 & 4\\ 1 & -6 & -4\end{bmatrix} = \begin{bmatrix}4 & -9 & -8\\ -6 & -4 & 5\\ 5 & 0 & -4\end{bmatrix}$$

Exercise 2 Use partitioned matrices to find AB, where

$$A = \begin{bmatrix}1 & 2 & 1 & 0\\ 3 & 4 & 0 & 1\\ 1 & 0 & -1 & 1\\ 0 & 1 & -1 & 1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

We partition A and B as follows

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So,

$$C = AB = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

where,

$$C_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix}$$

$$C_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Therefore,

$$AB = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 4 & 5 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Exercise 3 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Find A^0, A^1, A^2,...,A^7. What is A^{2015}?

$$
\begin{aligned}
A^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
A^1 = A &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \\
A^2 = AA &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\
A^3 = A^2A &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
A^4 = A^3A &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \\
A^5 = A^4A &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\
A^6 = A^5A &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^0 \\
A^7 = A^6A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = A
\end{aligned}
$$

Notice that $A^{2015} = A^x$, where

$$2015 \equiv x (\text{mod } 6)$$

By definition, x is the remainder of $\frac{2015}{6}$. So, $x = 5$, and

$$A^{2015} = A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Exercise 4 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find a formula for A^n, $n \geq 1$, and verify it using mathematical induction.

$$
\begin{aligned}
A^1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
A^2 = AA &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\
A^3 = A^2A &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \\
A^4 = A^3A &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

To this point, we conjecture that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Let's prove it by mathematical induction. The base case, $n = 1$, is true. Our induction hypothesis is

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

We now start with the induction step,

$$A^k A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$$

Since

$$A^{k+1} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$$

follows from the induction hypothesis, then the proof is completed.

5 More on matrices

Exercise 1 Use mathematical induction to prove that

$$(A^r)^T = (A^T)^r \quad r \geq 0$$

Base case:

$$\begin{aligned} I &= I \\ I^T &= I \\ (A^0)^T &= (A^T)^0 \end{aligned}$$

Induction hypothesis:

$$(A^k)^T = (A^T)^k$$

Induction step:

$$\begin{aligned} (A^k)^T &= (A^T)^k \\ (A^k)^T A^T &= (A^T)^k A^T \\ (AA^k)^T &= (A^T)^{k+1} \\ (A^{k+1})^T &= (A^T)^{k+1} \end{aligned}$$

Exercise 2 Prove that $(A^{-1})^T = (A^T)^{-1}$.

Using the fact that $A^T B^T = (BA)^T$ and the definition of inverse:

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T \\ &= I^T \\ &= I \end{aligned}$$

and

$$\begin{aligned} (A^{-1})^T A^T &= (AA^{-1})^T \\ &= I^T \\ &= I \end{aligned}$$

Therefore, the inverse of A^T is $(A^{-1})^T$, i.e.,

$$(A^T)^{-1} = (A^{-1})^T$$

Exercise 3 Prove that if a symmetric matrix is invertible, its inverse is symmetric too.

Our premises are $A = A^T$ and A is invertible.

$$\begin{aligned} A &= A^T \\ A^{-1} &= (A^T)^{-1} \\ A^{-1} &= (A^{-1})^T \end{aligned}$$

The last step follows from the previous exercise. And that is the definition of symmetry. Thus, A^{-1} is symmetric.

Exercise 4 Prove that $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$ by mathematical induction.

Base case:

For $n = 1$, it is obviously true.

$$(A_1)^{-1} = (A_1)^{-1}$$

For $n = 2$,

$$
\begin{aligned}
(A_1A_2)^{-1}A_1A_2 &= I \\
(A_1A_2)^{-1}A_1A_2A_2^{-1} &= IA_2^{-1} \\
(A_1A_2)^{-1}A_1I &= A_2^{-1} \\
(A_1A_2)^{-1}A_1 &= A_2^{-1} \\
(A_1A_2)^{-1}A_1A_1^{-1} &= A_2^{-1}A_1^{-1} \\
(A_1A_2)^{-1}I &= A_2^{-1}A_1^{-1} \\
(A_1A_2)^{-1} &= A_2^{-1}A_1^{-1}
\end{aligned}
$$

Induction hypothesis.

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$$

Induction step, using $(A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$ (proved in the base case):

$$
\begin{aligned}
(A_1A_2\cdots A_k)^{-1} &= A_k^{-1}\cdots A_2^{-1}A_1^{-1} \\
A_{k+1}^{-1}(A_1A_2\cdots A_k)^{-1} &= A_{k+1}^{-1}A_k^{-1}\cdots A_2^{-1}A_1^{-1} \\
((A_1A_2\cdots A_k)A_{k+1})^{-1} &= A_{k+1}^{-1}A_k^{-1}\cdots A_2^{-1}A_1^{-1} \\
(A_1A_2\cdots A_kA_{k+1})^{-1} &= A_{k+1}^{-1}A_k^{-1}\cdots A_2^{-1}A_1^{-1}
\end{aligned}
$$

Exercise 5 Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Find A^{-1}. Using A^{-1}, solve $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, and $A\mathbf{x} = \mathbf{b}_3$. Finally, use Gauss-Jordan elimination to simultaneously solve the three systems.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}$$

So,

$$\mathbf{x} = A^{-1}\mathbf{b}_1 = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1/2 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{b}_2 = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{b}_3 = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

Using Gauss-Jordan elimination,

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 6 & 5 & 2 & 0 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 2 & -1 & 4 & -4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 1 & -1/2 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{-2R_2 + R_1} \begin{bmatrix} 1 & 0 & 4 & -5 & 6 \\ 0 & 1 & -1/2 & 2 & -2 \end{bmatrix}$$

6 Subspaces and friends

Exercise 1

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that satisfy the given property. Is S a subspace of $\mathbb{R}^2$?

(1)

$$x = 0 \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \; : \; x = 0 \} \quad .$$

Since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ respects the constraints, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$.
Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in S.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \quad .$$

Since $u_1 = v_1 = 0$,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 + v_2 \end{bmatrix} \quad ,$$

which respects the constraints. So, $\mathbf{u} + \mathbf{v} \in S$.
Finally, for any $c \in \mathbb{R}$,

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ cu_2 \end{bmatrix} \quad .$$

Therefore, $c\mathbf{u} \in S$. So, S is a subspace of $\mathbb{R}^2$.

(2)

$$y = 2x \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \ : \ y = 2x \} \quad .$$

$x = 0$ and $y = 0$ satisfy $y = 2x$. Therefore, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in S.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ 2u_1 \end{bmatrix} + \begin{bmatrix} v_1 \\ 2v_1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 2u_1 + 2v_1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 2(u_1 + v_1) \end{bmatrix} \quad .$$

Thus, $\mathbf{u} + \mathbf{v} \in S$.
Finally, for any $c \in \mathbb{R}$,

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c \begin{bmatrix} u_1 \\ 2u_1 \end{bmatrix} = \begin{bmatrix} cu_1 \\ c2u_1 \end{bmatrix} = \begin{bmatrix} cu_1 \\ 2(cu_1) \end{bmatrix} \quad .$$

Therefore, $c\mathbf{u} \in S$. So, S is a subspace of $\mathbb{R}^2$.

(3)

$$x \geq 0 \quad , \quad y \geq 0 \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \ : \ x \geq 0 \ , \ y \geq 0 \} \quad .$$

$x = 0$ and $y = 0$ satisfy both restrictions. Therefore, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in S.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \quad .$$

Since the sum of two nonnegative numbers is a nonnegative number, $\mathbf{u} + \mathbf{v} \in S$.
Finally, for any $c \in \mathbb{R}$,

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \quad .$$

Notice that c can be negative. Since u_1 and u_2 are nonnegative, cu_1 and cu_2 can be negative. Therefore, there are vectors $c\mathbf{u} \notin S$. So, S is not a subspace of $\mathbb{R}^2$.

(4)

$$xy \geq 0 \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \ : \ xy \geq 0\} \quad .$$

$x = 0$ and $y = 0$ satisfy the constraint. Therefore, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$.

There are vectors $\mathbf{u}, \mathbf{v} \in S$ such that $\mathbf{u} + \mathbf{v} \notin S$. For instance,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad .$$

Since $xy = 3(-1)$ is a nonnegative number, $\mathbf{u} + \mathbf{v} \notin S$. Therefore, S is not a subspace of $\mathbb{R}^2$.

Exercise 2

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that satisfy the given property. Is S a subspace of $\mathbb{R}^3$?

(1)

$$x = y = z \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ x = y = z\} \quad .$$

Since $x = y = z = 0$ respects the constraints, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be vectors in S.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_1 + v_1 \\ u_1 + v_1 \end{bmatrix}$$

Since all components of the sum are equal, $\mathbf{u} + \mathbf{v} \in S$.

Finally, for any $c \in \mathbb{R}$,

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c \begin{bmatrix} u_1 \\ u_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_1 \\ cu_1 \end{bmatrix} .$$

Since all components of the product are equal, $c\mathbf{u} \in S$. So, S is a subspace of $\mathbb{R}^3$.

(2)

$$z = 2x \ , \ y = 0 \ .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ z = 2x \ , \ y = 0 \} \ .$$

Since $x = y = z = 0$ respects the constraints, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be vectors in S.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \\ 2u_1 \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \\ 2v_1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 0 \\ 2u_1 + 2v_1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ 0 \\ 2(u_1 + v_1) \end{bmatrix}$$

The third component of the sum is twice the first component, and the second component is zero. Therefore, $\mathbf{u} + \mathbf{v} \in S$.

Finally, for any $c \in \mathbb{R}$,

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c \begin{bmatrix} u_1 \\ 0 \\ 2u_1 \end{bmatrix} = \begin{bmatrix} cu_1 \\ 0 \\ c2u_1 \end{bmatrix} = \begin{bmatrix} cu_1 \\ 0 \\ 2cu_1 \end{bmatrix} .$$

The third component of the product is twice the first component, and the second component is zero. Therefore, $c\mathbf{u} \in S$. So, S is a subspace of $\mathbb{R}^3$.

(3)

$$x - y + z = 1 \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ x - y + z = 1\} \quad .$$

$x = y = z = 0$ do not respect the constraint. Therefore, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$. Thus, S is not a subspace of $\mathbb{R}^3$.

(4)

$$|x - y| = |y - z| \quad .$$

Thus,

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ |x - y| = |y - z|\} \quad .$$

$x = y = z = 0$ respect the constraint. So, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$.

$\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$ are vectors in S.

$$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 6 \end{bmatrix} \quad .$$

Since $|10 - 10| \neq |10 - 6|$, $\mathbf{u} + \mathbf{v} \notin S$. Therefore, S is not a subspace of $\mathbb{R}^3$.

Exercise 3 Is $\mathbf{b} \in \mathcal{C}(A)$? Is $\mathbf{w} \in \mathcal{R}(A)$?

(1)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \ , \ \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \ , \ \mathbf{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$

For the first part, lets find a row-echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Since the system is consistent, $\mathbf{b}$ is linear combination of the column vectors of A. Namely, $\mathbf{b} \in \mathcal{C}(A)$.

For the second part, lets apply Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Since we cannot further reduce the augmented matrix, $\mathbf{w}$ is not a linear combination of the row vectors of A. Namely, $\mathbf{w} \notin \mathcal{R}(A)$.

(2)

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 2 & 4 & -5 \end{bmatrix}$$

For the first part, lets find a row-echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 4 & 0 \end{bmatrix} \xrightarrow{-R_1 + R_3} \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 7 & -1 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

We can see that the system is consistent. Thus, vector $\mathbf{b}$ is linear combination of the column vectors of A. Namely, $\mathbf{b} \in \mathcal{C}(A)$.

For the second part, lets apply a series of row operations:

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \\ 2 & 4 & -5 \end{bmatrix} \xrightarrow{-R_1 + R_3} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & 7 \\ 2 & 4 & -5 \end{bmatrix} \xrightarrow{-2R_1 + R_4} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & 7 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{-R_2 + R_4} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the fourth row vector is a zero vector, $\mathbf{w}$ is a linear combination of the row vectors of A. Namely, $\mathbf{w} \in \mathcal{R}(A)$.

Exercise 4 Find a basis for $\mathcal{R}(A)$, $\mathcal{C}(A)$, and $\mathcal{N}(A)$.

(1)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Lets row-reduce this matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Since the system cannot be further reduced, the two row vectors are linearly independent (none of them was reduced to the zero vector). Therefore, $\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\}$ is a basis for $\mathcal{R}(A)$.

Since column vectors one and two are standard unit vectors, $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is a basis for $\mathcal{C}(A)$.

Now, lets find $\mathcal{N}(A)$:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

So,

$$\mathcal{N}(A) = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ y = -2z \ , \ x = z\} \quad .$$

In other words,

$$\mathcal{N}(A) = \{ \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} \} \quad = \{ z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \} \quad .$$

Thus, $\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \}$ is a basis for $\mathcal{N}(A)$.

(2)

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

Lets row-reduce this matrix:

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & 7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1/2 \\ 0 & -2 & 7 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 1/2 \\ 0 & -2 & 7 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 8 \end{bmatrix} \xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{7}{2}R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The three row vectors are linearly independent (none of them was reduced to the zero vector). Therefore, $\{\begin{bmatrix} 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 4 \end{bmatrix}\}$ is a basis for $\mathcal{R}(A)$.

Since the three column vectors are standard unit vectors, $\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-3\\1\\4\end{bmatrix}\}$ is a basis for $\mathcal{C}(A)$.

Now, lets find $\mathcal{N}(A)$:

$$\begin{bmatrix}1 & 1 & -3 & 0\\0 & 2 & 1 & 0\\1 & -1 & 4 & 0\end{bmatrix} \xrightarrow{GJ} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

So,

$$\mathcal{N}(A) = \{\begin{bmatrix}x\\y\\z\end{bmatrix} \ : \ x = 0 \ , \ y = 0 \ , \ z = 0\} \ .$$

In other words,

$$\mathcal{N}(A) = \{\begin{bmatrix}0\\0\\0\end{bmatrix}\} \ .$$

Thus, $\{\begin{bmatrix}0\\0\\0\end{bmatrix}\}$ is a basis for $\mathcal{N}(A)$.

(3)

$$A = \begin{bmatrix}2 & -4 & 0 & 2 & 1\\-1 & 2 & 1 & 2 & 3\\1 & -2 & 1 & 4 & 4\end{bmatrix}$$

Lets row-reduce this matrix:

$$\begin{bmatrix}2 & -4 & 0 & 2 & 1\\-1 & 2 & 1 & 2 & 3\\1 & -2 & 1 & 4 & 4\end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix}1 & -2 & 0 & 1 & 1/2\\-1 & 2 & 1 & 2 & 3\\1 & -2 & 1 & 4 & 4\end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix}1 & -2 & 0 & 1 & 1/2\\0 & 0 & 1 & 3 & 7/2\\1 & -2 & 1 & 4 & 4\end{bmatrix}$$

$$\xrightarrow{-R_1 + R_3} \begin{bmatrix}1 & -2 & 0 & 1 & 1/2\\0 & 0 & 1 & 3 & 7/2\\0 & 0 & 1 & 3 & 7/2\end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix}1 & -2 & 0 & 1 & 1/2\\0 & 0 & 1 & 3 & 7/2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

From the row-reduced form, we infer that $\{\begin{bmatrix}2 & -4 & 0 & 2 & 1\end{bmatrix}, \begin{bmatrix}-1 & 2 & 1 & 2 & 3\end{bmatrix}\}$ is a basis for $\mathcal{R}(A)$, and $\{\begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\}$ is a basis for $\mathcal{C}(A)$.

Now, lets find $\mathcal{N}(A)$:

$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 & 0 \\ -1 & 2 & 1 & 2 & 3 & 0 \\ 1 & -2 & 1 & 4 & 4 & 0 \end{bmatrix} \xrightarrow{GJ} \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 3 & 7/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So,

$$\mathcal{N}(A) = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \ : \ x_1 = 2x_2 - x_4 - (1/2)x_5 \ , \ x_3 = -3x_4 - (7/2)x_5 \} \quad .$$

In other words,

$$\mathcal{N}(A) = \{ \begin{bmatrix} 2x_2 - x_4 - (1/2)x_5 \\ x_2 \\ -3x_4 - (7/2)x_5 \\ x_4 \\ x_5 \end{bmatrix} \} = \{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix} \} \quad .$$

Thus, $\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for $\mathcal{N}(A)$.

Exercise 5 Is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ a basis for $\mathbb{R}^4$?

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{-R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 + R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 + R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_4 + R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since all column vectors in the reduced row-echelon form are standard unitary vectors, the original four column vectors are a basis for $\mathbb{R}^4$.

Exercise 6 Is $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ a basis for $\mathbb{R}^4$?

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 + R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only three column vectors in the reduced row-echelon form are standard unitary vectors. Therefore, the original four column vectors are not a basis for $\mathbb{R}^4$.

7 Abstract algebra

Exercise 1 There is only one group G of order 4 where $x + x = e$ for all $e \in G$. Find its Cayley table.

Let $\langle G, + \rangle$ be the group we are looking for. Since $x + x = e$, its Cayley table has the following entries:

$+$	e	a	b	c
e	e	a	b	c
a	a	e		
b	b		e	
c	c			e

Since each element of G must be once for each row and column, the remaining entries must be as follows:

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Exercise 2 Find all the groups of order 4 (remove automorphisms). Use Cayley tables.

All Cayley tables must have the following entries:

+	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

Since $a + a$ cannot be a:

Table 1

+	e	a	b	c
e	e	a	b	c
a	a	b		
b	b			
c	c			

Table 2

+	e	a	b	c
e	e	a	b	c
a	a	c		
b	b			
c	c			

Table 3

+	e	a	b	c
e	e	a	b	c
a	a	e		
b	b			
c	c			

From Table 1, only the next table can be derived:

Table 1.1

+	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

From Table 2, only the next table can be derived:

Table 2.1

+	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

However, Tables 1.1 and 2.1 are equivalent if we interchange b and c. Namely, there is an automorphism between both groups.

Table 1.1.1

+	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Table 2.1.1

+	e	a	c	b
e	e	a	c	b
a	a	c	b	e
c	c	b	e	a
b	b	e	a	c

From Table 3, the next tables can be derived:

Table 3.1

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Table 3.2

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Because of the diagonal of identity elements, there is not an automorphism between Table 3.2 and any other table. Nevertheless, there is between Table 3.1 and the other tables, where a is interchanged by b or c.

Table 3.1.1

+	e	b	a	c
e	e	b	a	c
b	b	a	c	e
a	a	c	e	b
c	c	e	b	a

Since all possible combinations have been explored, there are only two groups of order four. Their Cayley tables are Table 3.1 and 3.2.