Test 1 - linear algebra

INAOE

2024

Exercise 1 (1 point)

Find the projection of $\mathbf{v}$ onto $\mathbf{u}$.

$$\mathbf{u} = \begin{bmatrix} 1/2 \\ 1/4 \\ -1/2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1/2 \\ -1/4 \\ -1/2 \end{bmatrix}$$

Exercise 2 (0.5 points)

Prove the following property of the vectors in $\mathbb{R}^n$ (c, d are scalars and $\mathbf{u}$, $\mathbf{v}$ are vectors).

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

Exercise 3 (0.5 points)

Prove the following property of the vectors in $\mathbb{R}^n$ (c, d are scalars and $\mathbf{u}$, $\mathbf{v}$ are vectors).

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

Exercise 4 (0.5 points)

Normalize the following vector: $\begin{bmatrix} 5 \\ -2 \\ 3 \\ 4 \end{bmatrix}$

Exercise 5 (0.5 points)

What is the meaning of the following expressions?

$$||\mathbf{u} \cdot \mathbf{v}||$$

$$(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$$

Exercise 6 (2 points)

A cube has four diagonals. Prove that none of them is perpendicular to the others.

Exercise 7 (1.5 points)
Solve the following system of equations using Gauss-Jordan elimination.

$$\begin{aligned} 2w + 3x - y + 4z &= 4 \\ w - x + z &= 1 \\ 3x - 4x + y - z &= 0 \end{aligned}$$

Exercise 8 (1.5 points)
Find the values of k for which the following system has (a) one solution, (b) zero solutions, and (c) infinite solutions.

$$\begin{aligned} 2kx + 6y &= -7k - 2 \\ -kx - 3y &= 2k \end{aligned}$$

Exercise 9 (2 points) Use Gauss-Jordan elimination to solve the following system on $\mathbb{Z}_3$ (the modulo 3 numbers).

$$\begin{aligned} x + y &= 1 \\ y + z &= 0 \\ x + z &= 1 \end{aligned}$$

Exercise 10 (0.5 points)
How does a nonsingular matrix is related to the concepts of null space and identity matrix?

Exercise 11 (0.5 points) Construct an inconsistent system of linear equations with more variables than equations (you can begin with an augmented matrix and then apply row operations).

Exercise 12 (0.5 points) Let

$$A = \begin{bmatrix} -1 & 1 & 0 & 7 & 9 \\ 1 & -2 & -2 & 8 & 7 \\ -2 & 3 & 1 & 2 & -4 \\ 1 & 6 & 6 & 2 & -8 \\ 9 & -2 & -5 & -6 & 0 \end{bmatrix}$$

be a nonsingular matrix and let $\mathbf{b}$ be the vector $[\mathbf{b}]_i = i + 1$ for $1 \leq i \leq 5$. How many solutions does $\mathcal{LS}(A, \mathbf{b})$ has?

Exercise 1 (1 point)

Find the projection of $\mathbf{v}$ onto $\mathbf{u}$.

$$\mathbf{u} = \begin{bmatrix} 1/2 \\ 1/4 \\ -1/2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1/2 \\ -1/4 \\ -1/2 \end{bmatrix}$$

Answer

$$\begin{aligned}
proj_{\mathbf{u}}(\mathbf{u}) &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{v} \\
&= \frac{(1/2)(1/2) + (1/4)(-1/4) + (-1/2)(-1/2)}{(1/2)(1/2) + (1/4)(1/4) + (-1/2)(-1/2)} \begin{bmatrix} 1/2 \\ 1/4 \\ -1/2 \end{bmatrix} \\
&= \frac{(1/4) + (-1/16) + (1/4)}{(1/4) + (1/16) + (1/4)} \begin{bmatrix} 1/2 \\ 1/4 \\ -1/2 \end{bmatrix} \\
&= \frac{7/16}{9/16} \begin{bmatrix} 1/2 \\ 1/4 \\ -1/2 \end{bmatrix} = \frac{7}{9} \begin{bmatrix} 1/2 \\ 1/4 \\ -1/2 \end{bmatrix} \\
&= \begin{bmatrix} 7/18 \\ 7/36 \\ -7/18 \end{bmatrix}
\end{aligned}$$

Exercise 2 (0.5 points)

Prove the following property of the vectors in $\mathbb{R}^n$ (c, d are scalars and $\mathbf{u}$, $\mathbf{v}$ are vectors).

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

Answer

$$\begin{aligned}
c(\mathbf{u} + \mathbf{v}) &= c\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) = c \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} c(u_1 + v_1) \\ c(u_2 + v_2) \\ \vdots \\ c(u_n + v_n) \end{bmatrix} \\
&= \begin{bmatrix} cu_1 + cv_1 \\ cu_2 + cv_2 \\ \vdots \\ cu_n + cv_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = c\mathbf{u} + c\mathbf{v}
\end{aligned}$$

Exercise 3 (0.5 points)

Prove the following property of the vectors in $\mathbb{R}^n$ (c, d are scalars and $\mathbf{u}$, $\mathbf{v}$ are vectors).

$$(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

Answer

$$(c+d)\mathbf{u} = (c+d)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (c+d)u_1 \\ (c+d)u_2 \\ \vdots \\ (c+d)u_n \end{bmatrix} = \begin{bmatrix} cu_1 + du_1 \\ cu_2 + du_2 \\ \vdots \\ cu_n + du_n \end{bmatrix}$$

$$= \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + d\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = c\mathbf{u} + d\mathbf{u}$$

Exercise 4 (0.5 points)

Normalize the following vector: $\mathbf{v} = \begin{bmatrix} 5 \\ -2 \\ 3 \\ 4 \end{bmatrix}$

Answer

$$||\mathbf{v}|| = \sqrt{5^2 + (-2)^2 + 3^2 + 4^2} = \sqrt{25 + 4 + 9 + 16} = \sqrt{54}$$

$$\frac{1}{||\mathbf{v}||}\mathbf{v} = \frac{1}{\sqrt{54}}\begin{bmatrix} 5 \\ -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{54} \\ -2/\sqrt{54} \\ 3/\sqrt{54} \\ 4/\sqrt{54} \end{bmatrix}$$

Exercise 5 (0.5 points)

What is the meaning of the following expressions?

$$||\mathbf{u} \cdot \mathbf{v}||$$

$$(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$$

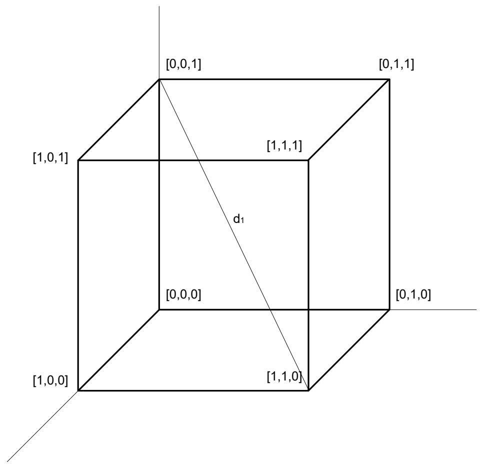
Answer The norm is defined for vectors, not scalars. Since $\mathbf{u} \cdot \mathbf{v}$ is a scalar, $||\mathbf{u} \cdot \mathbf{v}||$ makes no sense.

The dot product takes two vectors as argument. Since $\mathbf{u} \cdot \mathbf{v}$ is a scalar, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ makes no sense.

Exercise 6 (2 points)

A cube has four diagonals. Prove that none of them is perpendicular to the others.

Answer



$$\mathbf{d}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \qquad \mathbf{d}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\mathbf{d}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix} \qquad \mathbf{d}_4 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 1 \qquad \mathbf{d}_1 \cdot \mathbf{d}_3 = \begin{bmatrix}1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix} = -1$$

$$\mathbf{d}_1 \cdot \mathbf{d}_4 = \begin{bmatrix}1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\-1\end{bmatrix} = 1 \qquad \mathbf{d}_2 \cdot \mathbf{d}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix} = 1$$

$$\mathbf{d}_2 \cdot \mathbf{d}_4 = \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\-1\end{bmatrix} = -1 \qquad \mathbf{d}_3 \cdot \mathbf{d}_4 = \begin{bmatrix}-1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\-1\end{bmatrix} = 1$$

The dot product between all pairs of "diagonal" vectors is not zero. Therefore, they are not perpendicular to each other.

Exercise 7 (1.5 points)

Solve the following system of equations using Gauss-Jordan elimination.

$$\begin{aligned} 2w + 3x - y + 4z &= 4 \\ w - x + z &= 1 \\ 3x - 4x + y - z &= 0 \end{aligned}$$

Answer

$$\begin{bmatrix} 2 & 3 & -1 & 4 & 4 \\ 1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 2 & 3 & -1 & 4 & 4 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 5 & -1 & 2 & 2 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 5 & -1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 5 & -1 & 2 & 2 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 5 & -1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{-5R_2 + R_3} \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 4 & -3 & 2 \end{bmatrix} \xrightarrow{(1/4)R_3} \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3/4 & 1/2 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & 5/4 & 3/2 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3/4 & 1/2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & 5/4 & 3/2 \\ 0 & 1 & 0 & 1/4 & 1/2 \\ 0 & 0 & 1 & -3/4 & 1/2 \end{bmatrix}$$

Therefore,

$$S = \left\{ \begin{bmatrix} (3/2) - (5z/4) \\ (1/2) - (z/4) \\ (1/2) + (3z/4) \\ z \end{bmatrix} \quad : \quad z \in \mathbb{R} \right\}$$

Exercise 8 (1.5 points)

Find the values of k for which the following system has (a) one solution, (b) zero solutions, and (c) infinite solutions.

$$\begin{aligned} 2kx + 6y &= -7k - 2 \\ -kx - 3y &= 2k \end{aligned}$$

Answer

$$\begin{bmatrix} 2k & 6 & -7k-2 \\ -k & -3 & 2k \end{bmatrix} \xrightarrow{(1/2)R_1} \begin{bmatrix} k & 3 & (-7k-2)/2 \\ -k & -3 & 2k \end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix} k & 3 & (-7k-2)/2 \\ 0 & 0 & (-7k-2)/2+2k \end{bmatrix}$$

a) Since $r = 1$, the number of free variables is $n - r = 2 - 1 = 1$. Therefore, there are no values of k for which there is one solution.

b) There are zero solutions if and only if $(-7k-2)/2 + 2k \neq 0$, i.e., when $k \neq -2/3$.

c) There are infinite solutions if and only if $(-7k-2)/2+2k = 0$, i.e., when $k = -2/3$.

Exercise 9 (2 points) Use Gauss-Jordan elimination to solve the following system on $\mathbb{Z}_3$ (the modulo 3 numbers).

$$\begin{aligned} x + y &= 1 \\ y + z &= 0 \\ x + z &= 1 \end{aligned}$$

Answer

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{2R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{2R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where,

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Thus, $x = 1$, $y = 0$, and $z = 0$ is the solution.

Exercise 10 (0.5 points)

How does a nonsingular matrix is related to the concepts of null space and identity matrix?

Answer A square matrix A is nonsingular if and only if $\mathcal{LS}(A, \mathbf{0})$ has only the trivial solution; in other words, if its null space $\mathcal{N}(A)$ has only the zero vector. Besides, a nonsingular matrix always row-reduces to the identity matrix.

Exercise 11 (0.5 points) Construct an inconsistent system of linear equations with more variables than equations (you can begin with an augmented matrix and then apply row operations).

Answer

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 1 & -2 & -1 & 0 \\ 2 & 4 & 6 & 8 & 4 \end{bmatrix}$$

The systems of equations that correspond to each augmented matrix are equivalent. Since the first system is inconsistent, the following system is inconsistent too.

$$\begin{aligned} w + 2x + 3y + 4z &= 1 \\ w + x - 2y - z &= 0 \\ 2w + 4x + 6y + 8z &= 4 \end{aligned}$$

Exercise 12 (0.5 points) Let

$$A = \begin{bmatrix} -1 & 1 & 0 & 7 & 9 \\ 1 & -2 & -2 & 8 & 7 \\ -2 & 3 & 1 & 2 & -4 \\ 1 & 6 & 6 & 2 & -8 \\ 9 & -2 & -5 & -6 & 0 \end{bmatrix}$$

be a nonsingular matrix and let $\mathbf{b}$ be the vector $[\mathbf{b}]_i = i + 1$ for $1 \leq i \leq 5$. How many solutions does $\mathcal{LS}(A, \mathbf{b})$ has?

Answer Nonsingular matrices row-reduce to the identity matrix. Therefore, no matter what the vector $\mathbf{b}$ is, they always have one solution.