Test 2 - linear algebra

INAOE

2024

Exercise 1 (2 points)
Find A^{2015}.

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Exercise 2 (1 point)
Find X. Assume all the matrices involved are invertible.

$$(B^{-1}A^{-1}X)^{-1} = A(B^{-3}A)^{-1}$$

Exercise 3 (1.5 points)
Find A^{-1} using Gauss-Jordan elimination.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{on} \quad \mathbb{Z}_3$$

Exercise 4 (2.4 points)
Find a basis for $\mathcal{C}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$.

$$A = \begin{bmatrix} 2 & -4 & 0 & 1 & 1 \\ -1 & 2 & -1 & -4 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

Exercise 5 (1 points)
Is $\mathbf{b}$ in $\mathcal{C}(A)$? Is $\mathbf{w}$ in $\mathcal{R}(A)$?

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{w} = \begin{bmatrix} 2 & 4 & -5 \end{bmatrix}$$

Exercise 6 (0.5 points)

Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{Z}_3^3$?

Exercise 7 (2 points)

If possible, express matrix A as a product of elementary matrices.

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 5 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Extra:

Exercise 8 (0.5 point) What is a group?

Exercise 9 (0.5 point) What is a ring?

Exercise 10 (0.5 point) What is a field?

Exercise 1 (2 points)
Find A^{2015}.

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Answer (1 point)

$$\begin{aligned}
A^1 = A &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\
A^2 &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\
A^4 &= \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
A^5 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\
A^6 &= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
A^7 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\
A^8 &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Since $2015 \equiv 7 \pmod 8$, $A^{2015} = A^7$. Namely,

$$A^{2015} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Exercise 2 (1 point)
Find X. Assume all the matrices involved are invertible.

$$(B^{-1}A^{-1}X)^{-1} = A(B^{-3}A)^{-1}$$

Answer

$$
\begin{aligned}
(B^{-1}A^{-1}X)^{-1} &= A(B^{-3}A)^{-1} \\
X^{-1}(A^{-1})^{-1}(B^{-1})^{-1} &= AA^{-1}(B^{-3})^{-1} \\
X^{-1}AB &= (B^{-3})^{-1} \\
X^{-1}AB &= ((B^{3})^{-1})^{-1} \\
X^{-1}AB &= B^{3} \\
X^{-1}ABB^{-1} &= B^{2}BB^{-1} \\
X^{-1}AI &= B^{2}I \\
X^{-1}A &= B^{2} \\
X^{-1}AA^{-1} &= B^{2}A^{-1} \\
X^{-1}I &= B^{2}A^{-1} \\
X^{-1} &= B^{2}A^{-1} \\
(X^{-1})^{-1} &= (B^{2}A^{-1})^{-1} \\
X &= (A^{-1})^{-1}(B^{2})^{-1} \\
X &= AB^{-2}
\end{aligned}
$$

Exercise 3 (1.5 points)

Find A^{-1} using Gauss-Jordan elimination.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{on} \quad \mathbb{Z}_3$$

Answer

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{2R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{2R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

where,

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Exercise 4 (2.4 points)

Find a basis for $\mathcal{C}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$.

$$A = \begin{bmatrix} 2 & -4 & 0 & 1 & 1 \\ -1 & 2 & -1 & -4 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 1 \\ -1 & 2 & -1 & -4 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & -4 & 0 & 1 & 1 \\ 0 & 0 & -1 & -7/2 & 7/2 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1 + R_3} \begin{bmatrix} 2 & -4 & 0 & 1 & 1 \\ 0 & 0 & -1 & -7/2 & 7/2 \\ 0 & 0 & 1 & 7/2 & 7/2 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3} \begin{bmatrix} 2 & -4 & 0 & 1 & 1 \\ 0 & 0 & -1 & -7/2 & 7/2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \ , \ -R_2 \ , \ \frac{1}{7}R_3} \begin{bmatrix} 1 & -2 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 7/2 & -7/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the (almost) row-reduced form, we find that

$$\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \} \text{ is a basis for } \mathcal{C}(A).$$

$$\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 7/2 \\ -7/2 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T \} \text{ is a basis for } \mathcal{R}(A).$$

Since

$$\mathcal{N}(A) = \{ \begin{bmatrix} 2x_2 - (1/2)x_4 - (1/2)x_5 \\ x_2 \\ -(7/2)x_4 + (7/2)x_5 \\ x_4 \\ 0 \end{bmatrix} \ : \ x_2, x_4 \in \mathbb{R}\} = \{ \begin{bmatrix} 2x_2 - (1/2)x_4 \\ x_2 \\ -(7/2)x_4 \\ x_4 \\ 0 \end{bmatrix} \ : \ x_2, x_4 \in \mathbb{R}\}$$

$$= \{x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 1 \\ 0 \end{bmatrix} \ : \ x_2, x_4 \in \mathbb{R}\} \quad ,$$

$\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 1 \\ 0 \end{bmatrix} \}$ is a basis for $\mathcal{N}(A)$.

Exercise 5 (1 points)

Is $\mathbf{b}$ in $\mathcal{C}(A)$? Is $\mathbf{w}$ in $\mathcal{R}(A)$?

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{w} = \begin{bmatrix} 2 & 4 & -5 \end{bmatrix}$$

Answer

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 4 & 0 \end{array}\right] \xrightarrow{-R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 7 & -1 \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

Since the system is consistent, $\mathbf{b} \in \mathcal{C}(A)$.

$$\left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \\ \hline 2 & 4 & -5 \end{array}\right] \xrightarrow{-2R_1 + R_4} \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \\ \hline 0 & 2 & 1 \end{array}\right] \xrightarrow{-R_2 + R_4} \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Since the last row is a zero vector, $\mathbf{w} \in \mathcal{R}(A)$.

Exercise 6 (0.5 points)

Is $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$ a basis for $\mathbb{Z}_3^3$?

Answer

$$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\xrightarrow{2R_1+R_2}\begin{bmatrix}1&0&1\\0&1&2\\0&1&1\end{bmatrix}\xrightarrow{2R_2+R_3}\begin{bmatrix}1&0&1\\0&1&2\\0&0&2\end{bmatrix}$$

$$\xrightarrow{2R_3}\begin{bmatrix}1&0&1\\0&1&2\\0&0&1\end{bmatrix}\xrightarrow{R_3+R_2}\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{2R_3+R_1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Since the matrix row-reduces to the identity, the column vectors are a basis for $\mathbb{Z}_3^3$.

Exercise 7 (2 points)

If possible, express matrix A as a product of elementary matrices.

$$A=\begin{bmatrix}5&2&0\\5&4&0\\3&0&1\end{bmatrix}$$

Answer

$$\begin{bmatrix}5&2&0\\5&4&0\\3&0&1\end{bmatrix}\xrightarrow{-R_1+R_2}\begin{bmatrix}5&2&0\\0&2&0\\3&0&1\end{bmatrix}\xrightarrow{\frac{1}{5}R_1}\begin{bmatrix}1&2/5&0\\0&2&0\\3&0&1\end{bmatrix}\xrightarrow{\frac{1}{2}R_2}\begin{bmatrix}1&2/5&0\\0&1&0\\3&0&1\end{bmatrix}$$

$$\xrightarrow{-3R_1+R_3}\begin{bmatrix}1&2/5&0\\0&1&0\\0&-6/5&1\end{bmatrix}\xrightarrow{-\frac{2}{5}R_2+R_1}\begin{bmatrix}1&0&0\\0&1&0\\0&-6/5&1\end{bmatrix}\xrightarrow{\frac{6}{5}R_2+R_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The corresponding elementary matrices of each step are:

$$E_1=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix},\quad E_2=\begin{bmatrix}1/5&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad E_3=\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1\end{bmatrix}$$

$$E_4=\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix},\quad E_5=\begin{bmatrix}1&-2/5&0\\0&1&0\\0&0&1\end{bmatrix},\quad E_6=\begin{bmatrix}1&0&0\\0&1&0\\0&6/5&1\end{bmatrix}$$

By Gauss-Jordan elimination,

$$E_6E_5E_4E_3E_2E_1A=I$$

By definition,

$$A=(E_6E_5E_4E_3E_2E_1)^{-1}$$

By the shoes and socks theorem,

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} \quad ,$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad E_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad , \quad E_5 = \begin{bmatrix} 1 & 2/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6/5 & 1 \end{bmatrix}$$

Exercise 8 (0.5 point) What is a group?

Answer A group is an algebraic structure $\langle G, + \rangle$, where

- G is a set of elements,
- $+$ is a binary operation,
- $(a + b) + c = a + (b + c)$ (associativity),
- $a + b = c$ (closure),
- $a + e = a$ (unique identity element),
- $a + (-a) = e$ (inverse),
- $a, b, c \in G$.

Exercise 9 (0.5 point) What is a ring?

Answer A ring is an algebraic structure $\langle G, +, \times \rangle$, where

- $\langle G, + \rangle$ is a commutative group,
- $\times$ is a binary operation,
- $(a \times b) \times c = a \times (b \times c)$ (associativity),
- $a \times b = c$ (closure),
- $a \times e = a$ (identity),
- $a \times (b + c) = (a \times b) + (a \times c)$ (left-distributivity),
- $(b + c) \times a = (b \times a) + (c \times a)$ (right-distributivity).

Exercise 10 (0.5 point) What is a field?

Answer A field is an algebraic structure $\langle G, +, \times \rangle$, where

- $\langle G, +, \times \rangle$ is a commutative ring,
- $a \times (-a) = e$ (inverse),
- only the additive identity has no inverse.